BIFURCATION TO TRAVELING WAVES IN THE CUBIC-QUINTIC COMPLEX GINZBURG–LANDAU EQUATION

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Abstract. We consider the 1-dimensional complex Ginzburg–Landau equation (CGLE) which is a generic modulation equation describing the nonlinear evolution of patterns in fluid dynamics. The existence of a Hopf bifurcation from the basic solution was proved by Park [15]. We prove in this paper that the solution bifurcates to traveling waves which have constant amplitudes. We also prove that there exist kink-profile traveling waves which have variable amplitudes. The structure of the traveling waves is examined and it is proved by means of the center manifold reduction method and some perturbation arguments, that the variable amplitude traveling waves are quasi-periodic and they connect two constant amplitude traveling waves.

1. Introduction

In this paper we establish the existence of a bifurcation to traveling wave solutions of the 1-dimensional cubic quintic complex Ginzburg-Landau equation which reads

\[ \frac{\partial u}{\partial t} = \rho u + (1 + i\rho_0)u_{xx} + (1 + i\rho_1)|u|^2 u - (1 + i\rho_2)|u|^4 u, \]

where the unknown function \( u(x, t) \) is a complex-valued function. The parameters \( \rho_i (i = 0, 1, 2) \) are real numbers and \( \rho \) is the system parameter.

Existence and stability of periodic traveling waves of the type \( r \exp[i(ax + ct)] \) for the Ginzburg-Landau equation have been explored in the past few decades and we refer to Bernoff [3], Doelman [6], Doelman and Eckhaus [7], and Holmes [11].

The complex Ginzburg–Landau equation is a canonical model for weakly-nonlinear, dissipative systems and, for this reason, arises in a variety of settings, including fluid dynamics, nonlinear optics, chemical physics, mathematical biology, condensed matter physics, and statistical mechanics. In physics, this equation has long been an issue as a generic amplitude equation near the onset of instabilities that lead to chaotic dynamics in the theory of
phase transitions and superconductivity. For this reason it has had remarkable success in describing evolution phenomena in a broad range of physical systems, from physics to optics. More recently, it has been proposed and studied as a model for turbulent dynamics in nonlinear partial differential equations \cite{2}. In fluid dynamics the complex Ginzburg–Landau equation is found, for example, in the study of Poiseuille flow \cite{18}, the nonlinear growth of convection rolls in the Rayleigh–Bénard problem and Taylor–Couette flow \cite{13, 14, 17}. In this case, the bifurcation parameter $\rho$ plays the role of the Reynolds number.

Extensive mathematical studies have been conducted for the complex Ginzburg–Landau equation. We refer to Aranson and Kramer \cite{1} and Bartuccelli et al \cite{2} for hard turbulence, Choudhury \cite{4} for cubic quintic complex Ginzburg–Landau equation, Doering et al \cite{8} for low dimensional behavior, Ma, Park and Wang \cite{12} and Park \cite{15} for attractor bifurcation, and Doelman \cite{6} and Tang \cite{19} for traveling waves.

We consider (1.1) in a bounded interval $\Omega = (0, 2\pi)$ and assume that $u$ is the $\Omega$–periodic function. The most interesting case would be the complex Ginzburg-Landau equation which has only a stable cubic nonlinear term and gives us a supercritical bifurcation \cite{12}. In particular, Tang \cite{19} used the complex Gizburg-Landau equation of the cubic quintic type in which both cubic and quintic terms are stable. It has basically the same nature as the complex Ginzburg-Landau equation with only the stable cubic term. However, if the cubic nonlinear term becomes unstable, although some exceptions are known in \cite{16}, we need, in general, at least a stable quintic term to saturate an expected blowup of the solutions. This complex Ginzburg-Landau equation is called the cubic quintic complex Ginzburg-Landau equation and it has a subcritical bifurcation and a saddle node bifurcation. Two different phenomena have been explored in \cite{15}.

In this paper, we are interested in the bifurcation of solutions to (1.1) to a special type of traveling wave solution $u(x, t) = r \exp[i(ax + ct)]$ which satisfy $u_{\eta\eta} = -u$, where $\eta = x + ct$. The method has been applied to the Ginzburg-Landau equation by Tang \cite{19}.

The main ingredient of this paper is the center manifold reduction theorem and a perturbation method. The Center Manifold theorem is a useful tool as a reduction method. It reduces infinite dimensional PDE to finite dimensional ODE which can easily be dealt with. Early proofs of the Center Manifold Theorem only showed that center manifolds are $C^k \ (k \geq 1)$. However, a modern proof shows that they are $C^{k+1}$. Although this reduction eliminates some existing types of bifurcations occurring to solutions of the Ginzburg-Landau equation, it is good enough to use in this context.

Note that we will employ a perturbation method which is primarily due to Eckhaus \cite{9}. Eckhaus used a perturbation of the form

$$U(x, t) = \sum_{n=-\infty}^{\infty} U_n(t) \exp[i\mu_n x],$$
where $\mu_n = -\mu_{-n} = n\mu$ are wave numbers. He derived an infinite-dimensional dynamical system in terms of coefficients $U_n$ for $n = 0, \pm 1, \pm 2, \cdots$. Observing the decoupled system in $U_n$ and $U^*_{-n}$ in the linear approximation, he was able to analyze the stability of each mode. This analysis has been employed by many others and we refer to Doelman [6] and Holmes [11].

**Figure 1.1.** Bifurcation diagram to constant amplitude traveling waves $u_\pm$. The basic solution $u = 0$ is stable for $\rho < \frac{1}{c^2}$ but becomes unstable when $\rho > \frac{1}{c^2}$.

In this paper, we adopt the ideas and methods used in [6, 19] and apply them to the cubic quintic Ginzburg-Landau equation. We explore the existence of constant and variable amplitude quasi-periodic traveling waves and their structure. A constant amplitude means that the amplitude of a traveling wave does not change in time and space over a period. We can summarize our results as follows (see also Figure 1.1):

1. The basic solution $u = 0$ of the complex Ginzburg–Landau equation bifurcates to a constant amplitude traveling wave $u_-$ for $\frac{1}{c^2} - \frac{1}{4} < \rho < \frac{1}{c^2}$.
2. There exists a constant amplitude traveling wave $u_+$ for $\rho > \frac{1}{c^2} - \frac{1}{4}$ which extends to $\rho > \frac{1}{c^2}$. Due to [15], we may see that the bifurcated branch $u_+$ is a saddle node bifurcation.
3. There exists a kink-profile traveling wave $u(x, t)$ which has a variable amplitude and it connects two constant amplitude traveling waves $u_-$ and $u_+$: 
   \[ \lim_{t \to \pm\infty} u(x, t) = u_\pm. \]
4. The structure of the variable amplitude traveling wave is 
   \[ u(x, t) = r(x + \lambda t) \exp[-i\sigma t], \]
and \( u(x, t) \) is quasi-periodic.

The paper is organized as follows. We describe a bifurcation to the constant amplitude traveling waves in Section 2, followed by existence of variable amplitude traveling waves. Then, after introducing the Center Manifold theorem, we investigate the structure of the variable amplitude traveling waves in Section 3. Concluding remarks and some numerical bifurcation diagrams are given in Section 4.

2. Existence of Traveling Waves

2.1. Bifurcation to constant amplitude traveling waves. The purpose of this subsection is to derive the existence of constant amplitude traveling waves. Let \( u(x, t) = u_1(x, t) + iu_2(x, t) \). Then the equation (1.1) can be written as

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \rho u_1 + (u_1)_{xx} - \rho_0 (u_2)_{xx} + |u|^2 u_1 - \rho_1 |u|^2 u_2 - |u|^4 u_1 + \rho_2 |u|^4 u_2, \\
\frac{\partial u_2}{\partial t} &= \rho_0 u_1 - \rho (u_2)_{xx} + |u|^2 u_2 + \rho_1 |u|^2 u_1 - |u|^4 u_2 - \rho_2 |u|^4 u_1.
\end{align*}
\]

2.2. Take the polar coordinate system \( u(\eta) = r(\eta)e^{i\theta(\eta)} \).
Then the equation (2.3) becomes
\[ r'e^{i\theta} + i\theta' e^{i\theta} = \left( \rho - \frac{1}{c^2} - i\frac{\rho_0}{c^2} \right) r e^{i\theta} + (1 + i\rho_1)r^3 e^{i\theta} - (1 + i\rho_2)r^5 e^{i\theta}. \]
(2.4)

Therefore, we have
\[
\begin{cases}
  r' = \left( \rho - \frac{1}{c^2} \right) r + r^3 - r^5, \\
  \theta' = -\frac{\rho_0}{c^2} + \rho_1 r^2 - \rho_2 r^4.
\end{cases}
\]
(2.5)

Consider constant amplitude waves. Then we have
\[ (\rho - \frac{1}{c^2}) r + r^3 - r^5 = 0. \]
(2.6)

We can easily see that the roots of this algebraic equation are:
\[
\begin{align*}
r &= \begin{cases}
  0, & \text{if } \rho < \frac{1}{c^2} - \frac{1}{4}, \\
  0, \sqrt{\frac{1 \pm \sqrt{1 + 4(\rho - 1/c^2)}}{2}}, & \text{if } \frac{1}{c^2} - \frac{1}{4} \leq \rho < \frac{1}{c^2}, \\
  0, \sqrt{\frac{1 + \sqrt{1 + 4(\rho - 1/c^2)}}{2}}, & \text{if } \rho \geq \frac{1}{c^2}.
\end{cases}
\end{align*}
\]

Let
\[ r_\pm = \sqrt{\frac{1 \pm \sqrt{1 + 4(\rho - 1/c^2)}}{2}}. \]

From the second equation of (2.5), we have
\[ \theta_\pm = (-\frac{\rho_0}{c^2} + \rho_1 r_\pm^2 - \rho_2 r_\pm^4) \eta \]
which gives us constant amplitude traveling waves
\[
\begin{align*}
u(x, t) &= \begin{cases}
  0, & \text{if } \rho < \frac{1}{c^2} - \frac{1}{4}, \\
  0, r_\pm e^{i\theta_\pm}, & \text{if } \frac{1}{c^2} - \frac{1}{4} \leq \rho < \frac{1}{c^2}, \\
  0, r_+ e^{i\theta_+}, & \text{if } \rho \geq \frac{1}{c^2}.
\end{cases}
\end{align*}
\]

Linear stability and instability of \( u = 0 \) can easily be derived from the system (2.3). In fact, the jacobian matrix of (2.3) is
\[ J = \begin{pmatrix}
  \rho - \frac{1}{c^2} & \frac{\rho_0}{c^2} \\
  -\frac{\rho_0}{c^2} & \rho - \frac{1}{c^2}
\end{pmatrix}, \]
and its eigenvalues are
\[ \lambda(\rho) = \rho - \frac{1}{c^2} \pm \frac{\rho_0}{c^2}. \]
Moreover, it can also be proved that \( u = 0 \) is locally asymptotically stable for \( \rho < \frac{1}{c^2} \). The above arguments can be summarized in the following theorem:

**Theorem 2.1.** For complex Ginzburg–Landau equation under the space-periodic condition, the following assertions hold.

1. If \( \rho < \frac{1}{c^2} - \frac{1}{4} \), then there exists only steady state \( u = 0 \) and it is linearly stable.
2. If \( \frac{1}{c^2} - \frac{1}{4} \leq \rho < \frac{1}{c^2} \), then \( u = 0 \) bifurcates subcritically to a constant amplitude traveling wave solution \( u_- \) for \( \frac{1}{c^2} - \varepsilon \ll \rho < \frac{1}{c^2} \). There exists a saddle node bifurcation to a constant amplitude traveling wave solution \( u_+ \) for \( \frac{1}{c^2} - \frac{1}{4} < \rho \).
3. The bifurcated constant amplitude traveling wave \( u_+ \) extends to \( \rho \geq \frac{1}{c^2} \) and \( u = 0 \) becomes linearly unstable.

Here, the solution \( u(\eta) \) is a special type of wave solution which satisfies \( u_{\eta\eta} = -u \).

### 2.2. Existence of variable amplitude traveling waves.

Now we look for traveling waves \( u(x,t) = u(x + ct) = u(\eta) \) with limit conditions

\[
\lim_{\eta \to -\infty} |u(\eta)| = r_+ \quad \text{and} \quad \lim_{\eta \to \infty} |u(\eta)| = r_-
\]

where both \( r_\pm \) are constants. We follow that

1. \( u(\eta) \) are called kink-profile waves or wave fronts if \( r_+ \neq r_- \) which corresponds to a heteroclinic orbit of the system.
2. \( u(\eta) \) are called solitary waves or pulses, if \( r_+ = r_- \) and this type of waves corresponds to a homoclinic orbit.

**Theorem 2.2.**

1. For \( \rho > \frac{1}{c^2} \), we obtain a traveling wave \( u(\eta) \) which has a variable amplitude \( r(x,t) \). This traveling wave connects \( u = 0 \) and \( u = u_+ \).
2. For \( \frac{1}{c^2} - \frac{1}{4} < \rho < \frac{1}{c^2} \) and \( 0 < r^2 < r_-^2 \), we obtain a variable amplitude traveling wave which connects \( u = 0 \) and \( u = u_- \).
3. For \( \frac{1}{c^2} - \frac{1}{4} < \rho < \frac{1}{c^2} \) and \( r_-^2 < r^2 < r_+^2 \), we also obtain a variable amplitude traveling wave which connects two constant amplitude traveling waves \( u = u_- \) and \( u = u_+ \) (See picture 2.1).

**Proof.** From \( r' = (\rho - 1/c^2)r + r^3 - r^5 \), we have

\[
\frac{dr}{-r(r^2 - r_-^2)(r^2 - r_+^2)} = d\eta,
\]

where \( r_\pm \) are defined as above. At this step, we break it down into three cases labeled in Figure 2.1.

**Case I:** For \( \rho > \frac{1}{c^2} \), note that

\[
(2.7) \quad \frac{1}{-r(r^2 - r_-^2)(r^2 - r_+^2)} = A \frac{1}{r} + B \frac{1}{r + r_+} + C \frac{1}{r - r_+} + Dr + E \frac{1}{r^2 - r_-^2},
\]
Figure 2.1. Bifurcation diagram to constant amplitude traveling waves $u_{\pm}$.

where,

\[ A = \frac{1}{r_{-}^{2} r_{+}^{2}}, \quad B = C = \frac{1}{2r_{+}^{2} (r_{-}^{2} - r_{+}^{2})}, \quad D = \frac{-1}{r_{-}^{2} (r_{-}^{2} - r_{+}^{2})}, \quad E = 0. \]

Integrating (2.7) yields

\[ \frac{(r_{+}^{2} - r^{2})^{2}}{r^{2} (r_{-}^{2} - r_{+}^{2}) (r_{-}^{2} - r^{2}) r_{+}^{2}} = Ce^{\eta 2r_{+}^{2} r_{-}^{2} (r_{-}^{2} - r_{+}^{2})}, \]

where $C$ is a constant. Since $0 < r^{2} < r_{+}^{2}$,

\[ \begin{cases} r \rightarrow r_{+}, & \text{as } \eta \rightarrow \infty, \\
 r \rightarrow 0, & \text{as } \eta \rightarrow -\infty. \end{cases} \]

Therefore we have

\[ \lim_{\eta \rightarrow \infty} u(\eta) = u_{+}, \quad \lim_{\eta \rightarrow -\infty} u(\eta) = 0. \]
Figure 2.2. Case I with parameter values $c = 1$, $\rho = 1.5$ and $C = 1$ (left) and $c = 1$, $\rho = 1.1$ and $C = 1$ (right).

Case II: For $\frac{1}{c^2} - \frac{1}{4} < \rho < \frac{1}{c^2}$ and $0 < r^2 < r_-^2$, note that

\[
(2.8) \quad \frac{1}{-r(r^2 - r_-^2)(r^2 - r_+^2)} = \frac{A}{\rho} + \frac{B}{\rho + r_+} + \frac{C}{\rho - r_+} + \frac{D}{\rho + r_-} + \frac{E}{\rho - r_-},
\]

where,

\[
A = \frac{-1}{r_-^2 r_+^2}, \quad B = C = \frac{1}{r_-^2 r_+^2}, \quad D = \frac{1}{r_-^2 (r_+^2 + r_-^2)}, \quad E = 0.
\]

Integrating (2.8) yields

\[
\frac{(r^2 - r_-^2)^2}{r^2 (r^2 - r_-^2)(r_+^2 - r_-^2)r^2} = Ce^\eta 2r_+^2 r_-^2 (r_+^2 - r_-^2),
\]

where $C$ is a constant. Since $0 < r^2 < r_-^2$,

\[
\begin{cases}
  r \to 0, & \text{as } \eta \to \infty, \\
  r \to r_-, & \text{as } \eta \to -\infty.
\end{cases}
\]

Therefore we have

\[
\lim_{\eta \to \infty} u(\eta) = 0, \quad \lim_{\eta \to -\infty} u(\eta) = u_-.
\]
Case III: For $\frac{1}{c^2} - \frac{1}{4} < \rho < \frac{1}{c^2}$ and $r_2^- < r^2 < r_2^+$, note that we have the same partial fraction results as Case II, which yields

$$\frac{(r^2 - r_2^\pm)(r_2^+ - r^2)}{r^2(r_2^+ - r^2)(r_2^+ - r^2)} = Ce^{\eta 2r^2(r_2^+ - r_2^-)},$$

where $C$ is a constant. Since $r_2^- < r^2 < r_2^+$,

$$\begin{cases} r \to r_+, & \text{as } \eta \to \infty, \\ r \to r_-, & \text{as } \eta \to -\infty. \end{cases}$$

Therefore we have

$$\lim_{\eta \to \infty} u(\eta) = u_+, \quad \lim_{\eta \to -\infty} u(\eta) = u_-.$$
Since

\[ r' = -r(r^2 - r^2_-)(r^2 - r^2_+) \neq 0, \]

\( u(\eta) \) has a variable amplitude \( r(x, t) \). In each case, the variable amplitude traveling wave

\[ u(x, t) = r(x, t)e^{i\theta(x, t)} \]

connects two constant amplitude traveling waves \( u = 0 \) and \( u = u_+ \), \( u = 0 \) and \( u = u_- \) and \( u = u_+ \).

The proof is complete. \( \square \)

3. Structure of the Variable Amplitude Traveling Waves

In Section 3, we will investigate the structure of the variable amplitude traveling wave which connects two constant amplitude traveling waves. The main ingredient of the study is the center manifold reduction theorem and it is summarized in Section 3.1.

3.1. Center manifold theorem. Consider a nonlinear ordinary differential equation having an equilibrium point at the origin, say

\[ \frac{dx}{dt} = f(x), \]

with \( f : \mathbb{R}^n \to \mathbb{R}^n \), \( f(0) = 0 \). Assume that \( f \) is a vector field in \( C^{k+1} \) (\( k \geq 1 \)). Let \( V^c \) be the subspace of \( \mathbb{R}^q \) spanned by all generalized eigenvectors of \( Df(0) \) corresponding to eigenvalues with real part zero. The Center Manifold Theorem states that there exists \( \delta > 0 \) and a local center manifold \( M \) such that

1. There exists a \( C^k \) function \( \Phi : V^c \to \mathbb{R}^n \) such that

\[ \mathcal{M} = \{(x, \Phi(x)); x \in V^c, |x| < \delta \}. \]

2. The manifold \( \mathcal{M} \) is locally invariant and tangent to \( V^c \) at the origin.

3. Every globally bounded orbit remaining in a suitably small neighborhood of the origin is entirely contained inside \( \mathcal{M} \).

4. If \( x(t) \) is a solution of the equation,

\[ ||x(t) - \Phi(x(t))|| \leq ke^{-\delta t} \]

for some \( k > 0 \) and \( \delta > 0 \).

In essence, it says that near the origin, all interesting dynamics take place on an invariant manifold \( \mathcal{M} \), tangent to the center subspace \( V^c \). Its main utility lies in this dimensional reduction: instead of studying a flow on the entire space \( \mathbb{R}^n \), one can then restrict the analysis to a “center manifold” having the same dimension as \( V^c \).
3.2. **Bifurcation to quasi-periodic waves.** The bifurcation to the traveling waves was already obtained so that we assume that the equation (1.1) possesses a family of traveling waves of the form:

\[ u(x, t) = r \exp[i(ax + ct)]. \]

Inserting \( u(x, t) \) into (1.1) yields

\begin{align*}
    r^4 - r^2 + a^2 &= \rho, \\
    c + \rho_0 a^2 - \rho_1 r^2 + \rho_2 r^4 &= 0.
\end{align*}

Now we may assume that the amplitude \( r \), wavenumber \( a \) and frequency \( c \) satisfy the condition (3.1) and denote the values \( r_0, a_0 \) and \( c_0 \).

Consider perturbations of the form

\[ U(x, t) = \sum_{n=-\infty}^{\infty} U_n(t) \exp[i\mu_n x], \]

where \( \mu_n = -\mu_{-n} = n\mu \) are wavenumbers. Set

\begin{align*}
    u(x, t) &= [r_0 + U(x, t)] \exp[i(a_0 x + c_0 t)] \\
    &= r_0 \exp[i(a_0 x + c_0 t)] + \sum_{n=-\infty}^{\infty} U_n(t) \exp[i(a_0 + \mu_n) x + c_0 t].
\end{align*}
Inserting (3.3) into the equation (1.1) yields an infinite-dimensional dynamical system:

\[
\frac{dU_n}{dt} = \rho U_n - ic_0 U_n - (1 + i\rho_0)(a_0 + \mu_n)^2 U_n \\
+ (1 + i\rho_1)(2r_0^2 U_n + r_0^2 U^*_n) \\
- (1 + i\rho_2)(3r_0^2 U_n + 2r_0^4 U^*_n) \\
- (1 + i\rho_1)r_0 \sum_{k_1+k_2=n} (U_{k_1} U_{k_2} + 2U_{k_1} U^*_{k_2}) \\
- (1 + i\rho_1) \sum_{k_1+k_2+k_3=n} U_{k_1} U_{k_2} U^*_{k_3} \\
- (1 + i\rho_2)r_0^3 \sum_{k_1+k_2=n} (3U_{k_1} U_{k_2} + 6U_{k_1} U^*_{k_2} + U^*_{k_1} U^*_{k_2}) \\
- (1 + i\rho_2)r_0^3 \sum_{k_1+k_2+k_3=n} (U_{k_1} U_{k_2} U_{k_3} + 5U_{k_1} U_{k_2} U^*_{k_3} + U_{k_1} U^*_{k_2} U^*_{k_3}) \\
- (1 + i\rho_2)r_0^3 \sum_{k_1+k_2+k_3=n} U_{k_1} U_{k_2} U^*_{k_3} \\
- (1 + i\rho_2)r_0 \sum_{k_1+k_2+k_3+k_4=n} (U_{k_1} U_{k_2} U_{k_3} U^*_{k_4} + U_{k_1} U_{k_2} U^*_{k_3} U^*_{k_4}) \\
- (1 + i\rho_2) \sum_{k_1+k_2+k_3+k_4=n} U_{k_1} U_{k_2} U_{k_3} U^*_{k_4} U^*_{k_5},
\]

where \( n = 0, \pm 1, \pm 2, \cdots \) and the asterisk denotes the complex conjugate.

Linearization of this set of equations gives

\[
\frac{dU_n}{dt} = [\rho - ic_0 - (1 + i\rho_0)(a_0 + \mu_n)^2 + 2(1 + i\rho_1)r_0^2 - 3(1 + i\rho_2)r_0^2] U_n \\
+ [(1 + i\rho_1)r_0^2 - 2(1 + i\rho_2)r_0^4] U^*_n.
\]

Note that on the above linearization, the equations for \( U_n \) and \( U^*_n \) decouple into pairs of equations for \( (U_n(t), U^*_n(t)) \) for each \( n \). This is known as the sideband structure. Taking a general pair \( (U_n, U^*_n) \) with the wavenumber \( \mu_n \) and frequency \( c_0 \) yields

\[
\begin{align*}
\frac{dU_n}{dt} &= \Theta_n U_n + R_0 U^*_n, \\
\frac{dU^*_n}{dt} &= R_0^* U_n + \Theta^*_n U^*_n,
\end{align*}
\]

where

\[
\Theta_n = \rho - ic_0 - (1 + i\rho_0)(a_0 + \mu_n)^2 + 2(1 + i\rho_1)r_0^2 - 3(1 + i\rho_2)r_0^2,
\]

\[
R_0 = (1 + i\rho_1)r_0^2 - 2(1 + i\rho_2)r_0^4.
\]
Now we state and prove the main result of this paper.

**Theorem 3.1.** Assume that the amplitude $r_0$, wavenumber $a_0$ and frequency $c_0$ satisfy the condition (3.1). Assume also that the amplitude $|r_0| > \frac{1}{\sqrt{2}}$. Then for perturbations of the type (3.2), the bifurcated variable amplitude traveling wave, connecting two constant amplitude traveling waves are quasi-periodic. Moreover, the structure of the quasi-periodic traveling wave is

$$u(x, t) = r(x + \lambda t) \exp[-i\sigma t]$$

**Proof.** Since there exists only the trivial wave $u = 0$ for $\rho \leq \frac{1}{c^2} - \frac{1}{4}$, we may be concerned with the problem only for $\rho > \frac{1}{c^2} - \frac{1}{4}$. Consider the eigenvalue problem for the decoupled linear approximation,

$$\begin{vmatrix} \Theta - \lambda & R_0 \\ R_0^* & \Theta^*_{-n} - \lambda \end{vmatrix} = 0.$$  

For $n = 0$, the first mode, we have

$$\lambda^2 - (2\rho - 2a_0 + 4r_0^2 - 6r_0^4)\lambda$$

$$+ (\rho - a_0^2 + 2r_0^2 - 3r_0^4)^2 + (c_0 + \rho_0a_0^2 - 2\rho_0r_0^2 + 3\rho - 2r_0^4)^2$$

$$- (r_0^2 - 2r_0^4)^2 - (\rho_1r_0^2 - 2\rho_2r_0^4)^2 = 0.$$  

Due to (3.1), we have two eigenvalues,

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 2r_0^2 - 4r_0^4.$$  

Since $r_0^2 > \frac{1}{2}$, the first $n = 0$ mode is always stable.

For $n \neq 0$, the eigenvalue problem becomes

$$\lambda^2 - (2r_0^2 - 4r_0^4 - 2\rho_0^2 - 2\mu_n^2)\mu_n$$

$$+ (1 + \rho_0^2)\mu_n^4 + 4\rho_0a_0\mu_n^3$$

$$- 2(r_0^2 - 2r_0^4 + \rho_0\rho_1r_0^2 - 2\rho_0\rho_2r_0^4 + 2\rho_0a_0^2)\mu_n^2$$

$$- 4\rho_0a_0(r_0^2 - 2r_0^4)\mu_n$$

$$+ i4\rho_0(\rho_1r_0^2 - 2\rho_2r_0^4 - \rho_0\mu_n^2)\mu_n = 0.$$  

Note that the basic solution is stable if both eigenvalues $\lambda_1$ and $\lambda_2$ have a negative real part and is destabilized when one or both of the $\lambda_1$ cross the imaginary axis. To ensure stability, we need

$$2r_0^2 - 4r_0^4 - 2\mu_n^2 < 0,$$

as well as

$$\mathcal{D}_n > 0$$

where,

$$\mathcal{D}_n = (1 + \rho_0^2)\mu_n^4 + 4\rho_0a_0\mu_n^3$$

$$- 2(r_0^2 - 2r_0^4 + \rho_0\rho_1r_0^2 - 2\rho_0\rho_2r_0^4 + 2\rho_0a_0^2 + 2\rho_0a_0^2)\mu_n^2$$

$$- 4\rho_0a_0(r_0^2 - 2r_0^4)\mu_n.$$
It can be easily seen that the first inequality is always true for $r_0^2 > \frac{1}{2}$. However, we can find $\mu_c$, the critical wavenumber such that $D_1 < 0$ when $\mu_1 < \mu_c$. Therefore, we can see that the first unstable mode can be obtained for $n = 1$. Let $\lambda_1$ and $\lambda_{-1}$ be eigenvalues corresponding to $U_1$ and $U_{-1}$. Assume that $\text{Re}\lambda_1 > 0$ and $\text{Re}\lambda_{-1} < 0$ after $\mu_1$ crosses the threshold value $\mu_c$. From $r_0^2 - 2r_0^4 - 2\mu_1^2 < 0$ and the assumption on $\text{Re}\lambda_1$ and $\text{Re}\lambda_{-1}$, we can get

$$|\text{Re}\lambda_1| > |\text{Re}\lambda_{-1}|.$$ 

To be able to apply the center manifold theorem, we need to find a new set of basis $\{e_1, e_{-1}\}$ for the linear space generated by $\{U_1, U_{-1}\}$.

By the Center Manifold Theorem, there exist center manifold functions $\Phi_1$ and $\Phi_n$ given by

$$e_{-1} = \Phi_1(e_1) \quad \text{and} \quad U_n = \Phi_n(e_n).$$

We know that the center manifold functions involve only higher order terms,

$$\Phi_1 = o(|e_1|) \quad \text{and} \quad \Phi_n = o(|e_1|^n),$$

so that the governing equation becomes

$$\frac{de_1}{dt} = \lambda_1 e_1 + (1 + i\rho_3)|e_1|^2 e_1 - (1 + i\rho_4)|e_1|^4 e_1 + o(|e_1|^5).$$

Here, $\rho_3$ and $\rho_4$ are re-scaled scalars.

Since it can be shown that $\text{Im}\lambda_1 \neq 0$ at $\mu_n = \mu_1$, a Hopf bifurcation takes place and it creates the limiting cycle of the nonlinear system for $\mu_1 - \varepsilon < \mu_n < \mu_1$ where $|\varepsilon| \ll 1$:

$$|e_1|^2 = \frac{1 + \sqrt{1 + 4\text{Re}\lambda_1}}{2},$$

$$\arg e_1 = (\lambda_1 + \rho_3|e_1|^2 - \rho_4|e_1|^4)t + \text{terms in } e_1.$$

Therefore, the periodic solution of the equation (3.7) has an asymptotical expression,

$$e_1(t) = \gamma_1 \exp[i\omega_1 t + (|\gamma_1|^5)],$$

where, $\omega_1 = \lambda_1 + \rho_3|e_1|^2 - \rho_4|e_1|^4$, and $\gamma_1$ is a constant. We can see that it bifurcates from the plane wave.

Since $U_n(t)$ are governed by the center manifold function $\Phi_n$ which are higher order terms, we can get

$$\arg U_n(t) = n\omega_1 t + \text{terms in } e_1 \in U_1 + \text{higher order perturbation}.$$

System (3.4) should have periodic solutions of the form

$$U_n(t) = \gamma_n \exp[in\ell t] \quad (n = 0, \pm 1, \pm 2, \ldots),$$
for some constant \( \gamma_n \) and \( \ell \), which yield

\[
U(x, t) = \sum_{n=-\infty}^{\infty} \gamma_n \exp[i(n\ell t + \mu_n x)]
\]

\[
= \sum_{n=-\infty}^{\infty} \gamma_n \exp[in(\ell t + \mu x)]
\]

\[
= f(\mu x + \ell t),
\]

where \( f \) is a periodic function in \( \mu x + \ell t \). We may conclude that the structure of the bifurcated solution is

\[
u(x, t) = [r_0 + f(\mu x + \ell t)] \exp[i(a_0 x + c_0 t)]
\]

\[
= f[\mu(x + \frac{\ell}{\mu} t)] \exp[ia_0(x + \frac{\ell}{\mu} t)] \exp[i(c_0 - \frac{a_0 \ell}{\mu})t]
\]

\[
= r(x + \lambda t) \exp[-i\sigma t],
\]

which is quasi periodic in \( x + \lambda t \).

The proof is complete. \( \square \)

**Remark 3.2.** For \( \rho_0 = \rho_1 = \rho_2 = 0 \), the same analysis can be performed and it yields steady states since the frequency \( c_0 = 0 \) in (3.1). For the first mode \( n = 0 \), the eigenvalues are \( \lambda = 0, 2(r_0^2 - 2r_0^4) \). In this \( n = 0 \) mode, \( 0 \) is always an eigenvalue because we are linearizing along periodic solutions. The first unstable mode can also be obtained from \( n = 1 \). The necessary condition for that is \( a_0^2 > (2r_0^4 - r_0^2)/2 \) and \( \rho > -\frac{3}{2}r_0^2 \).

### 4. Concluding Remarks

In this paper we have studied the complex Ginzburg-Landau equation which has an unstable cubic term and a stable quintic term:

\[
\frac{\partial u}{\partial t} = \rho u + (1 + i\rho_0)u_{xx} + (1 + i\rho_1)|u|^2u - (1 + i\rho_2)|u|^4u.
\]
Without loss of generality, the equation has been normalized so that the coefficients of the linear and nonlinear dissipation (damping) terms are unity. The complex Ginzburg–Landau equation has a long history as a generic amplitude equation derived asymptotically near the onset of instabilities in fluid dynamical systems. The case with real coefficients was first derived by Newell and Whitehead [13], and Segel [17] to describe Bénard convection. The case with complex coefficients was put forth in a general setting by Newell and Whitehead [14] and DiPrima, Eckhaus, and Segel [5], and was shown by Stewartson and Stuart [18] to apply to plane Poiseuille flow.

Throughout this paper we have examined the bifurcation to traveling waves and their structure. Two types of traveling waves were obtained. The first type was constant amplitude traveling waves. The basic solution $u = 0$ bifurcates subcritically to a traveling wave which has a constant amplitude $r_-$ as the system parameter crosses 0. There was another bifurcation which is called a saddle node bifurcation. We proved that the saddle node bifurcation to a constant amplitude traveling wave, of which amplitude is $r_+$, occurs when $\rho > -\frac{1}{4}$ and this solution extends to $\rho > 0$. The arguments were obtained using Hopf bifurcation analysis. It was proved that the bifurcated waves $u_-$ is unstable and $u_+$ is stable in [15] and the diagram supports the arguments (see Figure 4.1).

If we take the initial amplitude $|r_0| < \frac{1}{2}$, the solution converges to the basic solution so that $u = 0$ is linearly stable. However, if it is taken outside, it converges to the outer traveling wave. We also obtained a kink profile traveling wave $u(x, t) = r(x, t) \exp[i\theta(x, t)]$ and $r(x, t)$ is a variable amplitude. It was proved that $u(x, t)$ approaches to $u_-(x, t)$ as $t \to -\infty$ and $u(x, t)$ approaches $u_+(x, t)$ as $t \to +\infty$. The structure of the variable amplitude
has also been examined. We used the Center Manifold Theorem and some perturbation analysis. In the perturbation analysis, we found that the first mode \( n = 0 \) is always stable and the first unstable mode occurs when \( n = 1 \). This analysis originates from Eckhaus [9]. The perturbed solutions are of the form \( u(x, t) = r(x + \lambda t) \exp[-i\sigma t] \) and are quasi-periodic solutions.

The complex Ginzburg–Landau equation can be obtained by perturbing the nonlinear Schrödinger equation (NLSE)

\[
\frac{\partial u}{\partial t} = i\rho_0 u_{xx} + i\rho_1 |u|^2 u,
\]

which is both Hamiltonian and integrable. Its extension, the cubic-quintic Schrödinger equation

\[
\frac{\partial u}{\partial t} = i\rho_0 u_{xx} + i\rho_1 |u|^2 u + i\rho_2 |u|^4 u,
\]

is Hamiltonian but not integrable. Due to the relations between the complex Ginzburg–Landau equation and NLSE, we can discuss the bifurcation to traveling waves of NLSE as an analogue to the relation of the driven Navier-Stokes equations to the Euler equations in fluid dynamics. It will be reported elsewhere.

**References**


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