DYNAMIC BIFURCATION OF THE GINZBURG–LANDAU EQUATION

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Abstract. We study in this article the bifurcation and stability of the solutions of the Ginzburg–Landau equation, using a notion of bifurcation called attractor bifurcation. We obtain in particular a full classification of the bifurcated attractor and the global attractor as \( \lambda \) crosses the first critical value of the linear problem. Bifurcations from the rest of the eigenvalues of the linear problem are obtained as well.

1. Introduction

In this article, we consider the bifurcation of attractors and invariant sets of the complex Ginzburg–Landau (GL) equation, which reads

\[
\frac{\partial u}{\partial t} - (\alpha + i\beta) \Delta u + (\sigma + i\rho)|u|^2u - \lambda u = 0,
\]

where the unknown function \( u : \Omega \times [0, \infty) \to \mathbb{C} \) is a complex-valued function and \( \Omega \subset \mathbb{R}^n \) is an open, bounded, and smooth domain in \( \mathbb{R}^n \) (\( 1 \leq n \leq 3 \)). The parameters \( \alpha, \beta, \sigma, \rho, \) and \( \lambda \) are real numbers and

\[
\alpha > 0, \quad \sigma > 0.
\]

The initial condition for (1.1) is given by

\[
\tag{1.3} u(x, 0) = \phi + i\psi.
\]

Also, (1.1) is supplemented with either the Dirichlet boundary condition,

\[
\tag{1.4} u|_{\partial \Omega} = 0,
\]

or the periodic boundary condition,

\[
\tag{1.5} \Omega = (0, 2\pi)^n \text{ and } u \text{ is } \Omega\text{-periodic}.
\]

The GL equation is an important equation in a number of scientific fields. It is directly related to the GL theory of superconductivity. In this context, the unknown function is the order parameter, the constants \( \beta \) and \( \rho \) are usually zero, and the bifurcation parameter \( \lambda \) is the GL parameter; see [11] and the references therein.

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In fluid dynamics the GL equation is found, for example, in the study of Poiseuille flow, the nonlinear growth of convection rolls in the Rayleigh–Benard problem and Taylor–Couette flow. In this case, the bifurcation parameter \( \lambda \) plays the role of a Reynolds number. The equation also arises in the study of chemical systems governed by reaction-diffusion equations.

There are extensive studies from the mathematical point of view for the GL equation, and we refer in particular to [2, 3, 1, 5, 6, 7, 11] and the references therein for studies related to the global attractors, inertial manifolds, and soft and hard turbulences described by the GL equations.

We study in this article the bifurcation and stability of the solutions of the complex GL equation. A nonlinear theory for this problem is established in this article using a notion of bifurcation called attractor bifurcation and its corresponding theorem developed recently by the authors in [9, 8]; see [10], a new book by two of the authors. The main objectives of this theory include

1. existence of bifurcation when the system parameter crosses some critical numbers,
2. dynamic stability of bifurcated solutions, and
3. the structure/patterns and their stability and transitions in the physical space.

More precisely, the main theorem associated with the attractor bifurcation states that as the control parameter crosses a certain critical value when there are \( m + 1 \) (\( m \geq 0 \)) eigenvalues across the imaginary axis, the system bifurcates from a trivial steady state solution to an attractor with dimension between \( m \) and \( m + 1 \), provided the critical state is asymptotically stable.

There are a few important features of the attractor bifurcation. First, the bifurcation attractor does not include the trivial steady state and is stable; hence it is physically important. Second, the attractor contains a collection of solutions of the evolution equation, including possibly steady states and periodic orbits as well as homoclinic and heteroclinic orbits. Third, it provides a unified point of view on dynamic bifurcation and can be applied to many problems in physics and mechanics. Fourth, from the application point of view, the Krasnoselskii–Rabinowitz theorem requires the number of eigenvalues \( m + 1 \) crossing the imaginary axis being an odd integer, and the Hopf bifurcation is for the case where \( m + 1 = 2 \). However, the new attractor bifurcation theorem obtained can be applied to cases for all \( m \geq 0 \). In addition, the bifurcated attractor, as mentioned earlier, is stable, which is another subtle issue for other known bifurcation theorems.

For the GL equation, bifurcation is obtained with respect to the parameter \( \lambda \), and the main results obtained can be summarized as follows.

First, for the GL equation with the Dirichlet boundary condition, let \( \lambda_1 \) be the first eigenvalue of the elliptic operator \(-\Delta\). Our main results in this case include the following.
1. If $\lambda \leq \alpha \lambda_1$, the trivial solution $u = 0$ is globally asymptotically stable. The global attractor of the GL equation consists exactly of the trivial steady state solution $u = 0$.

2. As $\lambda$ crosses $\alpha \lambda_1$, i.e., there exists an $\epsilon > 0$ such that for any $\alpha \lambda_1 < \lambda < \alpha \lambda_1 + \epsilon$, the GL problem bifurcates from the trivial solution an attractor $\Sigma_\lambda$. The bifurcated attractor $\Sigma_\lambda$ attracts the open set $L^2(\Omega, \mathbb{C})/\Gamma$, where $\Gamma$ is the stable manifold of $u = 0$ having codimension two in $L^2(\Omega, \mathbb{C})$.

More detailed structure of this bifurcated attractor can be classified as follows (see Figures 1.1 and 1.2.)

(a) If $|\beta| + |\rho| \neq 0$, then the bifurcated attractor consists of exactly one stable limiting cycle, i.e., $\Sigma_\lambda = S^1$, which is asymptotically stable. The global attractor $A_\lambda$ is a two-dimensional (2D) disk consisting of the stable limiting cycle $\Sigma_\lambda = S^1$, the (unstable) trivial steady state solution $u = 0$, and orbits connecting $\Sigma_\lambda = S^1$ and $u = 0$.

In particular, if $\beta \neq 0$, then the bifurcation is a Hopf bifurcation to a stable limiting cycle.

(b) If $\beta = \rho = 0$, then the bifurcated attractor $\Sigma_\lambda$ has dimension between 1 and 2 and is a limit of a sequence of 2D annulus $M_k$ with $M_{k+1} \subset M_k$, i.e., $\Sigma_\lambda = \cap_{k=1}^\infty M_k$.

Again in this case, the global attractor $A_\lambda$ is 2D, consisting of $\Sigma_\lambda$, $u = 0$ and the connecting orbits between them.  

![Figure 1.1. Bifurcation diagram for the GL equation with the Dirichlet boundary condition: (1) bifurcation appears at $\lambda = \alpha \lambda_1$, (2) bifurcated attractor $\Sigma_\lambda = S^1$ is the boundary of the shaded region, and (3) the global attractor $A_\lambda$ is the 2D disk, shown as the shaded region. Here the dotted line stands for the unstable trivial solution $u = 0$.](image-url)

Using a different method, we can in fact prove that $\Sigma_\lambda$ is also homeomorphic to $S^1$, which shall be reported elsewhere.
Second, for the GL equations equipped with the periodic boundary condition, similar results can be obtained as well. In particular, in the case where $|\beta| + |\rho| \neq 0$, we prove that the bifurcated attractor $\Sigma_\lambda$ is a sphere $S^1$, containing no steady state solutions, and the global attractor $A_\lambda$ is a 2 dimensional ball consisting of the trivial steady state $u = 0$, $\Sigma_\lambda$, and the orbits connecting them.

Finally, bifurcation from any eigenvalue of the Laplacian can also be obtained as for the first eigenvalue. It is worth mentioning that the complete structure of the global attractor for the bifurcations from the first eigenvalue is obtained, while no such information is available for bifurcations from the rest eigenvalues.

Important work on lower and upper bounds of the global attractor of the GL equation, together with their physical mechanisms, was done in the 1980’s in [2, 3, 1]. As mentioned earlier, the main objective of this article is to study bifurcation and transitions from the trivial solution. Hence we focus only on the local attractor near the trivial solution, which is part of the global attractor. Of course, near the first eigenvalue, complete information for both the global attractor and the bifurcated attractor is obtained in this article. For $\lambda$ near other eigenvalues, the results here demonstrate only the transitions of the trivial solution and provide some partial information on the low bounds of the global attractor. As far as the dimension of the global attractor is concerned, our results are consistent with the work in [2, 3, 1].

The paper is organized as follows. In Section 2, we recall the attractor bifurcation theory. Sections 3 and 4 study the bifurcation of the GL equations for the Dirichlet boundary condition and for the periodic boundary, respectively. Section 5 deals with bifurcation from the rest of the eigenvalues.
2. Abstract Bifurcation Theory

2.1. Preliminary. We recall in this section a general theory on attractor bifurcation for nonlinear evolution equations; see [9, 8].

Let $H$ and $H_1$ be two Hilbert spaces and $H_1 \hookrightarrow H$ be a dense and compact inclusion. We consider the nonlinear evolution equations

$$\begin{align*}
\frac{du}{dt} &= L_\lambda u + G(u, \lambda), \\
u(0) &= u_0,
\end{align*}$$

(2.1)

where $u : [0, \infty) \to H$ is the unknown function, $\lambda \in \mathbb{R}$ is the system parameter, and $L_\lambda : H_1 \to H$ are parameterized linear completely continuous fields depending continuously on $\lambda \in \mathbb{R}^1$, which satisfy

$$\begin{align*}
L_\lambda = -A + B_\lambda &\quad \text{is a sectorial operator,} \\
A : H_1 \to H &\quad \text{a linear homeomorphism,} \\
B_\lambda : H_1 \to H &\quad \text{the parameterized linear compact operators.}
\end{align*}$$

(2.3)

It is easy to see that $L_\lambda$ generates an analytic semigroup $\{e^{-tL_\lambda}\}_{t \geq 0}$. Then we can define fractional power operators $L_\lambda^\alpha$ for any $0 \leq \alpha \leq 1$ with domain $H_\alpha = D(L_\lambda^\alpha)$ such that $H_\alpha_1 \subset H_\alpha_2$ if $\alpha_1 > \alpha_2$, and $H_0 = H$.

Furthermore, we assume that the nonlinear terms $G(\cdot, \lambda) : H_\alpha \to H$ for some $1 > \alpha \geq 0$ are a family of parameterized $C^r$ bounded operators ($r \geq 1$) continuously depending on the parameter $\lambda \in \mathbb{R}^1$, such that

$$G(u, \lambda) = o(\|u\|_{H_\alpha}) \quad \forall \lambda \in \mathbb{R}^1.$$ 

(2.4)

Actually, in this paper we need only the following conditions for the operator $L_\lambda = -A + B_\lambda$, which ensure that $L_\lambda$ is a sectorial operator.

Let there be an eigenvalue sequence $\{\rho_k\} \subset \mathbb{C}$ and an eigenvector sequence $\{e_k, h_k\} \subset H_1$ of $A$:

$$\begin{align*}
Az_k &= \rho_k z_k, \quad z_k = e_k + ih_k, \\
\text{Re} \rho_k &\to +\infty \text{ as } k \to \infty, \\
|\text{Im} \rho_k/(\text{Re} \rho_k + a)| &\leq C \quad \text{for some constants } a, C > 0,
\end{align*}$$

(2.5)

such that $\{e_k, h_k\}$ is a basis of $H$.

Condition (2.5) implies that $A$ is a sectorial operator. Hence we can define fractional power operator $A^\alpha$ with domain $H_\alpha = D(A^\alpha)$. Then for the operator $B_\lambda : H_1 \to H$, we assume that there is a constant $0 \leq \theta < 1$ such that

$$B_\lambda : H_\theta \to H \text{ bounded } \forall \lambda \in \mathbb{R}.$$ 

(2.6)

Let $\{S_\lambda(t)\}_{t \geq 0}$ be an operator semigroup generated by the equation (2.1) which enjoys the following properties:

(i) For any $t \geq 0$, $S_\lambda(t) : H \to H$ is a linear continuous operator.

(ii) $S_\lambda(0) = I : H \to H$ is the identity on $H$ and

(iii) For any $t, s \geq 0$, $S_\lambda(t + s) = S_\lambda(t) \cdot S_\lambda(s)$. 

Let $\{S_\lambda(t)\}_{t \geq 0}$ be an operator semigroup generated by the equation (2.1) which enjoys the following properties:
Then the solution of (2.1) and (2.2) can be expressed as
\[ u(t) = S_\lambda(t)u_0, \quad t \geq 0. \]

**Definition 2.1.** A set \( \Sigma \subset H \) is called an invariant set of (2.1) if \( S(t)\Sigma = \Sigma \) for any \( t \geq 0 \). An invariant set \( \Sigma \subset H \) of (2.1) is called an attractor if \( \Sigma \) is compact, and there exists a neighborhood \( U \subset H \) of \( \Sigma \) such that for any \( \varphi \in U \) we have
\[ \lim_{t \to \infty} \text{dist}_H(u(t, \varphi), \Sigma) = 0. \]

The largest open set \( U \) satisfying (2.7) is called the basin of attraction of \( \Sigma \).

**Definition 2.2.**
1. We say that (2.1) bifurcates from \( (u, \lambda) = (0, \lambda_0) \) an invariant set \( \Omega_\lambda \) if there exists a sequence of invariant sets \( \{\Omega_{\lambda_n}\} \) of (2.1), \( \Omega_0 \subset \Omega_{\lambda_n}, \) such that for all \( \lambda \)
\[ \lim_{n \to \infty} \lambda_n = \lambda_0, \]
\[ \lim_{n \to \infty} \max_{x \in \Omega_{\lambda_n}} |x| = 0. \]

2. If the invariant sets \( \Omega_\lambda \) are attractors of (2.1), then the bifurcation is called attractor bifurcation.
3. If \( \Omega_\lambda \) are attractors and are homotopy equivalent to an \( m \)-dimensional sphere \( S^m \), then the bifurcation is called an \( S^m \)-attractor bifurcation.

### 2.2. Center Manifold Theorems.

We assume that the spaces \( H_1 \) and \( H \) can be decomposed into
\[ H_1 = E^1_1 \oplus E^2_1, \quad \dim E^1_1 < \infty, \quad \text{near } \lambda_0 \in \mathbb{R}^1, \]
\[ H = \tilde{E}^1_1 \oplus \tilde{E}^2_1, \quad \tilde{E}^1_1 = E^1_1, \quad \tilde{E}^2_1 = \text{closure of } E^2_1 \text{ in } H, \]
where \( E^1_1 \) and \( E^2_1 \) are two invariant subspaces of \( L_\lambda \); i.e., \( L_\lambda \) can be decomposed into \( L_\lambda = L^1_\lambda \oplus L^2_\lambda \) such that for any \( \lambda \) near \( \lambda_0 \),
\[ L^1_\lambda = L_\lambda|_{E^1_1} : E^1_1 \to \tilde{E}^1_1, \quad L^2_\lambda = L_\lambda|_{E^2_1} : E^2_1 \to \tilde{E}^2_1, \]
where all eigenvalues of \( L^1_\lambda \) possess negative real parts, and all eigenvalues of \( L^2_\lambda \) possess nonnegative real parts at \( \lambda = \lambda_0 \).

Thus, for \( \lambda \) near \( \lambda_0 \), (2.1) can be rewritten as
\[ \begin{cases} \frac{dx}{dt} = L^1_\lambda x + G_1(x, y, \lambda), \\ \frac{dy}{dt} = L^2_\lambda y + G_2(x, y, \lambda), \end{cases} \]
where \( u = x + y \in H_1, \ x \in E^1_1 \), \( y \in E^2_1 \), \( G_1(x, y, \lambda) = P_1G(u, \lambda) \), and \( P_1 : H \to \tilde{E}_1 \) are canonical projections. Furthermore, we let
\[ E^2_2(\alpha) = E^2_2 \cap H_\alpha, \]
with \( \alpha \) given by (2.4).
Theorem 2.3 (Center Manifold Theorem, [4]). Assume (2.4)–(2.6), (2.8), and (2.9). Then there exist a neighborhood of \( \lambda_0 \) given by \( |\lambda - \lambda_0| < \delta \) for some \( \delta > 0 \), a neighborhood \( B_\lambda \subset E_\lambda^1 \) of \( x = 0 \), and a \( C^1 \) function \( h(\cdot, \lambda) : B_\lambda \to E_\lambda^2(\alpha) \), depending continuously on \( \lambda \), such that

1. \( h(0, \lambda) = 0, D_x h(0, \lambda) = 0 \);
2. the set \( M_\lambda = \{ (x, y) \in H_1 \mid x \in B_\lambda, y = h(x, \lambda) \in E_\lambda^2(\alpha) \} \), called center manifold, is locally invariant for (2.1), i.e. for any \( u_0 \in M_\lambda \), \( u_\lambda(t, u_0) \in M_\lambda, \forall 0 \leq t < t_{u_0} \), for some \( t_{u_0} > 0 \), where \( u_\lambda(t, u_0) \) is the solution of (2.1); and
3. if \( (x_\lambda(t), y_\lambda(t)) \) is a solution of (2.10), then there are a \( \beta_\lambda > 0 \) and \( k_\lambda > 0 \) with \( k_\lambda \) depending on \( (x_\lambda(0), y_\lambda(0)) \) such that
\[
\|y_\lambda(t) - h(x_\lambda(t), \lambda)\|_{H} \leq k_\lambda e^{-\beta_\lambda t}.
\]

If we consider only the existence of the local center manifold, then conditions in (2.9) can be modified in the following fashion. Let the operator \( L_\lambda = L_\lambda^1 \oplus L_\lambda^2 \) and \( L_\lambda^2 \) be decomposed into
\[
\begin{align*}
L_\lambda^2 &= L_\lambda^2_1 \oplus L_\lambda^2_2, \\
E_\lambda^2 &= E_\lambda^2_1 \oplus E_\lambda^2_2, \tilde{E}_\lambda^2 = \tilde{E}_\lambda^2_1 \oplus \tilde{E}_\lambda^2_2, \\
\dim E_\lambda^2_1 &= \dim \tilde{E}_\lambda^2_1 < \infty,
\end{align*}
\]

\begin{align*}
\begin{cases}
L_\lambda^1: E_\lambda^1 \to \tilde{E}_\lambda^1 \\
L_\lambda^2_1: E_\lambda^2_1 \to \tilde{E}_\lambda^2_1 \\
L_\lambda^2_2: E_\lambda^2_2 \to \tilde{E}_\lambda^2_2
\end{cases}
\end{align*}

such that at \( \lambda = \lambda_0 \)

\begin{align*}
\begin{cases}
eigenvalues of L_\lambda^1: E_\lambda^1 \to \tilde{E}_\lambda^1 &\text{have zero real parts}, \\
eigenvalues of L_\lambda^2_1: E_\lambda^2_1 \to \tilde{E}_\lambda^2_1 &\text{have positive real parts}, \\
eigenvalues of L_\lambda^2_2: E_\lambda^2_2 \to \tilde{E}_\lambda^2_2 &\text{have negative real parts}.
\end{cases}
\end{align*}

Then we have the following center manifold theorem.

Theorem 2.4. Assume (2.4)–(2.6), (2.8), (2.11), and (2.12). Then the conclusions (1) and (2) in Theorem 2.3 hold true.

2.3. Attractor Bifurcation. A complex number \( \beta = \alpha_1 + i\alpha_2 \in \mathbb{C} \) is called an eigenvalue of \( L_\lambda : H_1 \to H \) if there are \( x, y \in H_1 \) such that
\[
L_\lambda z = \beta z, \quad z = x + iy,
\]
or, equivalently,
\[
L_\lambda x = \alpha_1 x - \alpha_2 y, \\
L_\lambda y = \alpha_2 x + \alpha_1 y.
\]
Now let the eigenvalues (counting the multiplicity) of \( L_\lambda \) be given by 
\[ \beta_1(\lambda), \beta_2(\lambda), \cdots, \beta_k(\lambda) \in \mathbb{C}. \]

Suppose that
\[
(2.13) \quad \text{Re} \beta_i(\lambda) = \begin{cases} 
< 0 & \text{if } \lambda < \lambda_0, \\
0 & \lambda = \lambda_0 \\
> 0 & \lambda > \lambda_0, 
\end{cases} \quad (1 \leq i \leq m + 1),
\]
\[
(2.14) \quad \text{Re} \beta_j(\lambda_0) < 0 \quad \forall \ m + 2 \leq j.
\]

Let the eigenspace of \( L_\lambda \) at \( \lambda_0 \) be
\[
E_0 = \bigcup_{i=1}^{m+1} \left\{ u \in H_1 \mid (L_{\lambda_0} - \beta_i(\lambda_0))^k u = 0, k = 1, 2, \ldots \right\}.
\]
It is known that \( \dim E_0 = m + 1. \)

Let \( H_1 = H = \mathbb{R}^n. \) The following attractor bifurcation theorem was proved in [9].

**Theorem 2.5 (Attractor Bifurcation Theorem).** Let \( H_1 = H = \mathbb{R}^n, \) the conditions (2.13) and (2.14) hold true, and \( u = 0 \) be a locally asymptotically stable equilibrium point of (2.1) at \( \lambda = \lambda_0. \) Then the following assertions hold true.

1. **Equation (2.1) bifurcates from \((u, \lambda) = (0, \lambda_0)\) an attractor \( A_\lambda \) for \( \lambda > \lambda_0, \) with \( m \leq \dim A_\lambda \leq m + 1, \) which is connected as \( m > 0. \)
2. **The attractor \( A_\lambda \) is a limit of a sequence of \((m + 1)\)-dimensional annulus \( M_k \) with \( M_{k+1} \subset M_k. \) In particular, if \( A_\lambda \) is a finite simplicial complex, then \( A_\lambda \) has the homotopy type of \( S^m. \)
3. **For any \( u_\lambda \in A_\lambda, \) \( u_\lambda \) can be expressed as \( u_\lambda = v_\lambda + o(\|v_\lambda\|_{H_1}), \) \( v_\lambda \in E_0. \)
4. **If \( G : H_1 \to H \) is compact and the equilibrium points of (2.1) in \( A_\lambda \) are finite, then we have the index formula**
\[
\sum_{u_i \in A_\lambda} \text{ind} \left[ -(L_\lambda + G), u_i \right] = \begin{cases} 
2 & \text{if } m = \text{even}, \\
0 & \text{if } m = \text{odd}.
\end{cases}
\]
5. **If \( u = 0 \) is globally stable for (2.1) at \( \lambda = \lambda_0, \) then for any bounded open set \( U \subset H \) with \( 0 \in U \) there is an \( \varepsilon > 0 \) such that as \( \lambda_0 < \lambda < \lambda_0 + \varepsilon, \) the attractor \( A_\lambda \) bifurcated from \((0, \lambda_0)\) attracts \( U/\Gamma \) in \( H, \) where \( \Gamma \) is the stable manifold of \( u = 0 \) with codimension \( m + 1. \) In particular, if (2.1) has a global attractor in \( H \) then \( U = H. \)**

**Remark 2.6.** As \( H_1 \) and \( H \) are infinite dimensional Hilbert spaces, if (2.1) satisfies the conditions (2.4)–(2.6),(2.13), and (2.14) and \( u = 0 \) is a locally (global) asymptotically stable equilibrium point of (2.1) at \( \lambda = \lambda_0, \) then the assertions (1)–(5) of Theorem 2.5 hold; see [9, 8].
3. Bifurcation of the GL Equation with Dirichlet Boundary Condition

As mentioned in the introduction, we study in this article attractor bifurcation of the GL equation under either the Dirichlet or the periodic boundary conditions.

We start with the GL equation with the Dirichlet boundary condition. Let

\[ H_k(\Omega, \mathbb{C}) = \{ u_1 + iu_2 \mid u_j \in H_k(\Omega), j = 1, 2 \}, \]
\[ H^1_0(\Omega, \mathbb{C}) = \{ u \in H^1(\Omega, \mathbb{C}) \mid u|_{\partial\Omega} = 0 \}, \]

where \( H^k(\Omega) \) is the usual real-valued Sobolev space.

Let \( \lambda_1 \) be the first eigenvalue of \(-\Delta\) with the Dirichlet boundary condition (1.4). Then we have the following main bifurcation theorem for the GL equations with the Dirichlet boundary condition.

Theorem 3.1. (1) If \( \lambda \leq \alpha \lambda_1 \), \( u = 0 \) is a globally asymptotically stable equilibrium point of (1.1) with (1.4).

(2) As \( \lambda \) crosses \( \alpha \lambda_1 \), i.e., for any \( \alpha \lambda_1 < \lambda < \alpha \lambda_1 + \epsilon \) for some \( \epsilon > 0 \), the problem (1.1) with (1.4) bifurcates from \( (u, \lambda) = (0, \alpha \lambda_1) \) an attractor \( \Sigma_\lambda \).

(3) The bifurcated attractor \( \Sigma_\lambda \) has dimension between 1 and 2 and is a limit of a sequence of 2D annulus \( M_k \) with \( M_{k+1} \subset M_k \); i.e., \( \Sigma_\lambda = \bigcap_{k=1}^{\infty} M_k \).

(4) If \( \beta \neq 0 \), then the bifurcation is a Hopf bifurcation, i.e., \( \Sigma_\lambda = S^1 \), which is asymptotically stable (limiting cycle).

(5) If \( \beta = 0 \) and \( \rho \neq 0 \), then the bifurcated attractor \( \Sigma_\lambda \) is a periodic orbit, which is a limiting cycle.

(6) Moreover, for each \( \alpha \lambda_1 < \lambda < \alpha \lambda_1 + \epsilon \), the bifurcated attractor \( \Sigma_\lambda \) attracts the open set \( L^2(\Omega, \mathbb{C})/\Gamma \), where \( \Gamma \) is the stable manifold of \( u = 0 \) having codimension two in \( L^2(\Omega, \mathbb{C}) \).

Proof. We proceed in several steps as follows.

Step 1. Let \( u = u_1 + iu_2 \). The GL problem (1.1) with (1.3) can be equivalently written as follows:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \alpha \Delta u_1 - \beta \Delta u_2 + \lambda u_1 - \sigma |u|^2 u_1 + \rho |u|^2 u_2, \\
\frac{\partial u_2}{\partial t} &= \beta \Delta u_1 + \alpha \Delta u_2 + \lambda u_2 - \sigma |u|^2 u_2 - \rho |u|^2 u_1, \\
ù_1(x, 0) &= \phi(x), \quad u_2(x, 0) = \psi(x).
\end{align*}
\]

We shall apply Theorems 2.3 and 2.5 to prove this theorem. Let

\[ H_1 = H^2(\Omega, \mathbb{C}) \cap H^1_0(\Omega, \mathbb{C}), \quad H = L^2(\Omega, \mathbb{C}). \]
The mappings $L_\lambda = -A + B_\lambda$ and $G : H_1 \to H$ are defined as

\[-Au = \left( \begin{array}{c} \alpha \Delta u_1 - \beta \Delta u_2 \\ \beta \Delta u_1 + \alpha \Delta u_2 \end{array} \right),\]
\[B_\lambda u = \lambda \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right),\]
\[Gu = \left( \begin{array}{c} -\sigma |u|^2 u_1 + \rho |u|^2 u_2 \\ -\sigma |u|^2 u_2 - \rho |u|^2 u_1 \end{array} \right).\]

It is known that $H_{1/2} = H_0^1(\Omega, \mathbb{C})$. By the Sobolev embedding theorems and $1 \leq n \leq 3$, the mapping $G : H_{1/2} \to H$ is $C^\infty$. The condition (2.4) is fulfilled.

Let $\{\lambda_k\} \subset \mathbb{R}$ and $\{e_k\} \subset H^2(\Omega) \cap H^1_0(\Omega)$ be the eigenvalues and eigenvectors of $-\Delta$ with the Dirichlet boundary condition (1.4)

\[-\Delta e_k = \lambda_k e_k,
\]
\[e_k|_{\partial \Omega} = 0.\]

We know that

\[0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lambda_k \to \infty \text{ as } k \to \infty,\]

and $\{e_k\}$ is an orthogonal basis of $L^2(\Omega)$.

It is easy to see that the eigenvalues of $A$ are given by \[\alpha \lambda_k \pm i \beta \lambda_k, \quad k = 1, 2, \cdots\]
with the corresponding eigenvectors \[z_k = e_k + i e_k,\]
and $\{e_k, i e_j | 1 \leq k, j < \infty\}$ is an orthogonal basis of $H$. Thus the conditions (2.5) and (2.6) are valid for $A$ and $B_\lambda$. The eigenvalues of $L_\lambda = -A + B_\lambda$ are as follows:

\[(3.2) \quad (\lambda - \alpha \lambda_k) \pm i \beta \lambda_k, \quad k = 1, 2, \cdots.\]

In addition, the spaces $H$ and $H_1$ can be decomposed into the form

\[H_1 = E_1 \oplus E_2 \quad \text{and} \quad H = E_1 \oplus \tilde{E}_2,\]
\[E_1 = \{x_1 e_1 + i y_1 e_1 | x_1, y_1 \in \mathbb{R}\},\]
\[E_2 = \{\sum_{k=2}^{\infty} (x_k + i y_k) e_k | \sum_{k=2}^{\infty} \lambda_k^2 (x_k^2 + y_k^2) < \infty\},\]
\[\tilde{E}_2 = \{\sum_{k=2}^{\infty} (x_k + i y_k) e_k | \sum_{k=2}^{\infty} (x_k^2 + y_k^2) < \infty\},\]

and the operator $L_\lambda$ is decomposed into

\[L_\lambda = \mathcal{L}_1^\lambda \oplus \mathcal{L}_2^\lambda,\]
\[\mathcal{L}_1^\lambda = L_\lambda|_{E_1} : E_1 \to E_1, \quad \mathcal{L}_2^\lambda = L_\lambda|_{E_2} : E_2 \to \tilde{E}_2.\]

Thus the conditions (2.8) and (2.9) are satisfied.
By the center manifold theorem, the attractor bifurcation of (1.1) with (1.4) is equivalent to that of the bifurcation equations

\[
\begin{align*}
\frac{dx_1}{dt} &= (\lambda - \alpha \lambda_1)x_1 + \beta \lambda_1 y_1 + P_1 G_1(x_1 + iy_1 + h), \\
\frac{dy_1}{dt} &= -\beta \lambda_1 x_1 + (\lambda - \alpha \lambda_1)y_1 + P_1 G_2(x_1 + iy_1 + h),
\end{align*}
\]

where \( h = h_1 + ih_2 \) is the center manifold function satisfying

\[ h(x_1, y_1) = o(|x_1| + |y_1|), \]

and \( P_1 G_i(u) (i = 1, 2) \) are given by

\[
\begin{align*}
P_1 G_1(u) &= \int_{\Omega} [-\sigma |u|^2 u_1 + \rho |u|^2 u_2] e_1 dx, \\
P_1 G_2(u) &= \int_{\Omega} [-\sigma |u|^2 u_2 - \rho |u|^2 u_1] e_1 dx,
\end{align*}
\]

\[ u = u_1 + iu_2 = \sum_{k=1}^{\infty} (x_k + iy_k) e_k. \]

**Step 2.** Now we show that \( u = 0 \) is a globally asymptotically stable equilibrium of (2.1) for \( \lambda \leq \alpha \lambda_1 \). In fact, from (3.1) we can derive that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 dx = \int_{\Omega} (-\alpha |\nabla u|^2 + \lambda |u|^2 - \sigma |u|^4) dx \\
\leq - \int_{\Omega} [((\alpha \lambda_1 - \lambda)|u|^2 + |u|^4] dx
\]

which implies that \( u = 0 \) is globally stable.

**Step 3.** We know that (1.1) has a global attractor; see [12]. Obviously, for the eigenvalues (3.2) of \( L_\lambda \), the conditions (2.13) and (2.14) for \( \lambda_0 = \alpha \lambda_1 \) are satisfied. Therefore, by Remark 2.6, (2.1) bifurcates from \( (u, \lambda) = (0, \alpha \lambda_1) \) an attractor \( \Sigma_\lambda \) which attracts \( H/\Gamma \).

**Step 4.** We now prove that \( \Sigma_\lambda = S^1 \).

Obviously, as \( \beta \neq 0 \) the bifurcation is the typical Hopf bifurcation. Therefore, we have to consider only the case where \( \beta = 0 \). In this case, the bifurcation equations (3.3) read

\[
\begin{align*}
\frac{dx}{dt} &= \varepsilon x + P_1 G_1(xe_1 + h_1 + iy_1 + ih_2), \\
\frac{dy}{dt} &= \varepsilon y + P_1 G_2(xe_1 + h_1 + iy_1 + ih_2),
\end{align*}
\]
where $\varepsilon = \lambda - \alpha \lambda_1 > 0$ sufficiently small. By (3.4) we have

\[
P_1 G_1(x, y) = \int_{\Omega} \left[ -\sigma u_1^3 + \rho u_2^3 - \sigma u_2^2 u_1 + \rho u_1^2 u_2 \right] e_1 dx
\]

\[
= (\text{by } h(x, y) = o(|x| + |y|))
\]

\[
= a(-\sigma x^3 + \rho y^3 - \sigma y^2 x + \rho x^2 y) + o(|x|^3 + |y|^3),
\]

where $u_1 = x e_1 + h_1(x, y)$, $u_2 = y e_1 + h_2(x, y)$, and $a = \int_{\Omega} e_1^4(x) dx > 0$.

Thus, the bifurcation equations (3.5) lead to

\[
\begin{aligned}
\frac{dx}{dt} &= \varepsilon x - a(\sigma x^3 - \rho y^3 + \sigma y^2 x - \rho x^2 y) + o(|x|^3 + |y|^3), \\
\frac{dy}{dt} &= \varepsilon y - a(\sigma y^3 + \rho x^3 + \sigma x^2 y + \rho y^2 x) + o(|x|^3 + |y|^3).
\end{aligned}
\]  

(3.6)

We can see that the attractor $\Sigma_\lambda$ has no nonzero singular point; i.e., the singular point $u = 0$ of (1.1) with (1.4) is unique provided $|\rho| + |\beta| \neq 0$, because from (3.1) we have

\[
\int_{\Omega} \left[ u_2 \frac{\partial u_1}{\partial t} - u_1 \frac{\partial u_2}{\partial t} \right] dx = \int_{\Omega} |\beta| |\nabla u|^2 + \rho |u|^4 dx.
\]

By Theorem 2.5, $\Sigma_\lambda$ has the homotopy type of $S^1$; hence $\Sigma_\lambda$ contains at least one periodic orbit provided $\rho \neq 0$.

Take the polar coordinate system

\[x = r \cos \theta, \quad y = r \sin \theta.\]

Then (3.6) becomes

\[
\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon r - \frac{a(\sigma r^3 - \rho y^3 + \sigma y^2 r - \rho x^2 y)}{a\rho r}, \\
r(0) &= r_0.
\end{aligned}
\]  

(3.7)

From (3.7) it follows that

\[
\frac{a\rho}{2}(r^2(2\pi) - r^2(0)) = \int_0^{2\pi} [\varepsilon - a\sigma r^2 + o(r^2)] d\theta.
\]

Because $r^2 = r^2(\theta, r_0)$ is $C^\infty$ on $r_0 \geq 0$, we have the Taylor expansion

\[r^2(\theta, r_0) = r_0^2 + R(\theta) \cdot o(|r_0|^2), \quad R(0) = 0.
\]

Hence we get

\[
\frac{a\rho}{2}(r^2(2\pi) - r^2(0)) = 2\pi \varepsilon - 2\pi a\sigma r_0^2 + o(|r_0|^2).
\]

Obviously the initial values $r_0 > 0$ in (3.7) satisfying

\[
2\pi \varepsilon - 2\pi a\sigma r_0^2 + o(|r_0|^2) = 0
\]  

(3.8)
are corresponding to the periodic orbits of (3.6). It is easy to see that the solution $r_0^2 > 0$ of (3.8) near $r_0 = 0$ is unique. Thus we deduce that $\Sigma_\lambda$ is a periodic orbit provided $\rho \neq 0$.

The proof is complete. □

4. Bifurcation of the GL Equation with Periodic Boundary Condition

For the GL equation with periodic boundary condition, the first eigenspace is larger than that in the Dirichlet boundary condition case, and to proceed, we need the following function spaces:

$$H^k_{\text{per}}(\Omega, \mathbb{C}) = \{ u \in H^k(\Omega, \mathbb{C}) | u \text{ satisfy (1.5)} \}.$$ 

Then the main result in this section is the following theorem.

**Theorem 4.1.** For the GL equation (1.1) with the periodic boundary condition (1.5), we have the following assertions.

(1) (a) As $\lambda > \alpha$, the problem (1.1) with (1.5) bifurcates from $(u, \lambda) = (0, \alpha)$ an invariant set $\Sigma_\lambda$. $\Sigma_\lambda$ has dimension between $4n - 1$ and $4n$ and is a limit of a sequence of $4n$ annulus $M_k$ with $M_{k+1} \subset M_k$; i.e., $\Sigma_\lambda = \bigcap_{k=1}^\infty M_k$.

(b) If $|\rho| + |\beta| \neq 0$, then $\Sigma_\lambda$ contains no steady state solutions of (1.1) with (1.5).

(2) (a) As $\lambda \leq 0$, $u = 0$ is globally asymptotically stable.

(b) As $\lambda > 0$ the problem (1.1) with (1.5) bifurcates from $(u, \lambda) = (0, 0)$ an attractor $\Sigma_\lambda \subset L^2(\Omega, \mathbb{C})$. The bifurcated attractor $\Sigma_\lambda$ has dimension between $1$ and $2$, and is a limit of a sequence of $2m$-dimensional annulus $M_k$ with $M_{k+1} \subset M_k$, i.e., $\Sigma_\lambda = \bigcap_{k=1}^\infty M_k$.

(c) If $\rho \neq 0$ then $\Sigma_\lambda$ is a periodic orbit.

(d) $\Sigma_\lambda$ attracts $L^2(\Omega, \mathbb{C})/\Gamma$, where $\Gamma$ is the stable manifold of $u = 0$ with codimension two in $L^2(\Omega, \mathbb{C})$.

**Proof.** Let $H_1 = H^2_{\text{per}}(\Omega, \mathbb{C})$, $H = L^2_{\text{per}}(\Omega, \mathbb{C})$ and the mappings $L_\lambda$ and $G : H_1 \to H$ be as defined in the previous section. Similar to the proof of Theorem 3.1, $L_\lambda$ and $G$ satisfy the conditions in Theorem 2.3.

We know that the eigenvalue problem

$$\begin{cases} -\Delta e_k = \lambda_k e_k, \\
e_k(x + 2k\pi) = e_k(x) \end{cases}$$

has an eigenvalue sequence

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lambda_k \to \infty \text{ as } k \to \infty,$$

and an eigenvector sequence $\{e_k\}$ which constitutes a common orthogonal basis of $H_1$ and $H$. The second eigenvalue $\lambda_1 = 1$ has multiplicity $2n$, i.e., $\lambda_1 = \cdots = \lambda_{2n}$, with the first eigenvectors

$$\sin x_j, \quad \cos x_j \quad (x = (x_1, \cdots, x_n) \in \Omega = (0, 2\pi)^n).$$
Eigenvalues of $L_\lambda$ are as in (3.2), and the second eigenvalue $\Lambda_1 = (\lambda - \alpha) \pm i\beta$ has multiplicity $4n$. For simplicity, let

$$e_{2j-1} = \sin x_j, \quad e_{2j} = \cos x_j, \quad (j = 1, \cdots, n).$$

Then the spaces $H$ and $H_1$ can be decomposed into the following form

$$H = E_1 \oplus \tilde{E}_2,$$

$$E_1 = \left\{ \sum_{j=1}^{2n} (z_{1j} + iz_{2j})e_j \mid z_{1j}, z_{2j} \in \mathbb{R} \right\},$$

$$\tilde{E}_2 = E_1^\perp.$$

Then the bifurcation equations of (1.1) with (1.5) are given by

$$\begin{align*}
\frac{dZ_1}{dt} &= (\lambda - \alpha)Z_1 + \beta Z_2 + PG_1(u), \\
\frac{dZ_2}{dt} &= -\beta Z_1 + (\lambda - \alpha)Z_2 + PG_2(u),
\end{align*}$$

where $u = u_1 + iu_2$ and

$$(u_1, u_2) = (Z_1 + h_1(Z_1, Z_2), Z_2 + h_2(Z_1, Z_2)),$$

$$(Z_1, Z_2) = \sum_{j=1}^{2n} (z_{1j}, z_{2j})e_j.$$

Here $h = h_1 + ih_2 : E_1 \to \tilde{E}_2$ is the center manifold function satisfying

$$h(Z_1, Z_2) = o(\|Z_1\| + \|Z_2\|)$$

and

$$PG_1(u) = \sum_{j=1}^{2n} e_j \int_{\Omega} \left[ -\sigma|u|^2u_1 + \rho|u|^2u_2 \right] e_j dx,$$

$$PG_2(u) = \sum_{j=1}^{2n} e_j \int_{\Omega} \left[ -\sigma|u|^2u_1 - \rho|u|^2u_2 \right] e_j dx.$$

By Theorem 2.5, we infer from (4.3) that the problem (1.1) and (1.5) bifurcates from $(u, \lambda) = (0, \alpha)$ an invariant set $\Sigma_\lambda$.

The proof is complete. \(\square\)

**Remark 4.2.** In fact, the invariant set $\Sigma_\lambda$ of (1.1) with (1.5) is a sphere $S^{4n-1}$; namely, $\Sigma_\lambda$ is homeomorphic to a sphere $S^{4n-1}$. The topological structure of an attractor of vector fields should be stable provided some nondegenerate conditions hold. we shall discuss the topic elsewhere.
5. Bifurcation of invariant sphere $S^m$

More generally, for the GL equations we have the bifurcation theorem of invariant sphere $S^m \ (m \geq 1)$ at any eigenvalue.

**Theorem 5.1.** Let $\lambda_k$ be an eigenvalue of $-\Delta$ with the boundary condition (1.4), or (1.5), which has multiplicity $m \geq 1$. Then, as $\lambda > \alpha \lambda_k$, the problem (1.1) with (1.4), or (1.1) with (1.5), bifurcates from $(u, \lambda) = (0, \alpha \lambda_k)$ an invariant set $\Sigma_\lambda$. This invariant set $\Sigma_\lambda$ has dimension between $2m - 1$ and $2m$ and is a limit of a sequence of $2m$ dimensional annulus $M_k$ with $M_{k+1} \subset M_k$; i.e., $\Sigma_\lambda = \cap_{k=1}^{\infty} M_k$. If $|\beta| + |\rho| \neq 0$, then there is no singular point in $\Sigma_\lambda$.

**Proof.** We denote the eigenvectors of $-\Delta$ corresponding to $\lambda_k$ by $\{e^*_1, \ldots, e^*_m\}$.

Thus, the space $H_1$ and $H$ defined in Theorem 3.1 or Theorem 4.1 can be decomposed into

$$H_1 = E_m \oplus E^\perp_m, \quad H = \tilde{E}_m \oplus \tilde{E}^\perp_m,$$

$$E_m = \text{span}\{e^*_i + ie^*_j \ | \ 1 \leq i, j \leq m\},$$

$$E^\perp_m = \{u \in H_1 \ | \ \langle u, v \rangle_{H_1} = 0 \ \forall v \in E_m\},$$

$$\tilde{E}_m = E_m, \quad \tilde{E}^\perp_m = \{u \in H \ | \ \langle u, v \rangle_H = 0 \ \forall v \in \tilde{E}_m\}.$$

By the center manifold theorem, the bifurcation equations of (1.1) with (1.4), or (1.1) with (1.5), at $\lambda = \lambda_k$ are equivalent to

$$\begin{align*}
\frac{\partial v_1}{\partial t} &= \alpha \Delta v_1 - \beta \Delta v_2 + \lambda v_1 + PG_1(v + h(v)), \\
\frac{\partial v_2}{\partial t} &= \beta \Delta v_1 + \alpha \Delta v_2 + \lambda v_2 + PG_2(v + h(v)),
\end{align*}$$

where $\lambda$ is near $\lambda_k$, $v = v_1 + iv_2 \in E_m$, and $h : E_m \rightarrow E^\perp_m$ is the center manifold function, $G = (G_1, G_2) : H_1 \rightarrow H$ defined as in Theorem 3.1 or Theorem 4.1, and $P : H \rightarrow \tilde{E}_m$ is the projection.

The equations (5.1) are a system of ordinary differential equations with order $2m$:

$$\begin{align*}
\frac{dZ_1}{dt} &= (\lambda - \alpha)Z_1 + \beta Z_2 + [-\sigma |Z|^2 Z_1 + \rho |Z|^2 Z_2] + o(|Z|^3), \\
\frac{dZ_2}{dt} &= -\beta Z_1 + (\lambda - \alpha)Z_2 + [-\sigma |Z|^2 Z_2 - \rho |Z|^2 Z_1] + o(|Z|^3),
\end{align*}$$

where $Z = Z_1 + iZ_2$. The eigenvalues of the linear part are still $(\lambda - \alpha \lambda_k) \pm i\beta \lambda_k$, with multiplicity $2m$. 
By Theorem 2.5 it suffices to prove that \( v = 0 \) is asymptotically stable for (5.2) at \( \lambda = \alpha \lambda_k \). For \( \lambda = \alpha \lambda_k \), we infer from (5.2) that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |v|^2 dx = \int_{\Omega} G(v + h(v))v dx = (\text{by } h(v) = o(|v|)) = \int_{\Omega} G(v)v dx + o(|v|^4) = -\sigma \int_{\Omega} |v|^4 dx + o(|v|^4),
\]

which implies that \( v = 0 \) is asymptotically stable for (5.2) at \( \lambda = \alpha \lambda_k \). The proof is complete.

\[\square\]

References


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