14.6 Directional Derivatives & the Gradient Vector.

* Directional derivatives enable us to find the change of rate of a function of two or more variables in any direction.

\[
\frac{df}{dx} : \text{rate of change in the direction of } x
\]

\[
\frac{df}{dy} : 
\]

**Def:**
The directional derivative of \( f \) at \((x_0, y_0)\) in the direction of a unit vector \( \mathbf{u} = (a, b) \) is

\[
D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}
\]

if this limit exists.
Ex. \( u = i = (1, 0) \Rightarrow D_uf = f_x \)
\( u = j = (0, 1) \Rightarrow D_n f = f_y \)

**Theorem**

If \( f \) is differentiable, then \( f \) has a directional derivative \( D_u f \) in the direction of any unit vector \( u = (a, b) \) and

\[ D_u f (x,y) = f_x (x,y) a + f_y (x,y) b \]

**Ex.** Find \( D_u f \) if \( f(x,y) = x^3 - 3xy + ay^2 \) and \( u \) is the unit vector given by angle \( \theta = \frac{\pi}{6} \). What is \( D_\theta f (1, 2) \)?

\[ u = (\cos \frac{\pi}{6}, \sin \frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2}) \]

So,
\[ D_u f (x,y) = (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 2y) \frac{1}{2} \]
\[ D_\theta f (1, 2) = (-3) \frac{\sqrt{3}}{2} + (13) \frac{1}{2} = \frac{13 - 3\sqrt{3}}{2} \]

**Q.**
1. In which direction does \( f \) change fastest?
2. What is the maximum rate of change?
The Gradient Vector

\[ D_u f(x,y) = f_x(x,y) a + f_y(x,y) b \]
\[ = (f_x, f_y) \cdot (a, b) \]
\[ = (f_x, f_y) \cdot w \]
\[ \text{gradient of } f \]

Def. \( f(x,y) \).
\[ \nabla f(x,y) = (f_x(x,y), f_y(x,y)) = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} \]
\[ \text{gradient of } f. \]
\[ S \ldots D_u f(x,y) = \nabla f \cdot w. \]

Ex. \( f(x,y) = x \sin(xy) \)
\[ \nabla f = (\sin(xy) + x^2 \cos(xy), x^2 \cos(xy)) \]

Ex. Find the directional derivative of \( f(x,y) = x^2y^3 - 4y \) at \( (2, -1) \) in the direction of \( w = 2i + 5j \)
\[ \nabla f(2, -1) = \left( \frac{\partial}{\partial x}(2xy^3), \frac{\partial}{\partial y}(3x^2y^2 - 4) \right)_{(2, -1)} = (2 \cdot 2(-1)^3, 3 \cdot 2^2(-1)^2 - 4) \]
\[ = (-4, 8) \]
\[ \nabla f(2, -1) \cdot (2, 5) = \frac{1}{\sqrt{29}} (-4, 8) \cdot (2, 5) = \frac{1}{\sqrt{29}} (-8 + 40) \]
\[ = \left( \frac{32}{\sqrt{29}} \right) \]
In the functions of three variables: \( f(x, y, z) \)

\[
\nabla f(x, y, z) = (f_x, f_y, f_z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k.
\]

\[
D_w f(x, y, z) = \nabla f(x, y, z) \cdot w.
\]

\[\text{Ex} \quad f(x, y, z) = x \sin y z\]

(a) Find the gradient of \( f \)

(b) Find the directional derivative of \( f \) at \( (1, 3, 0) \) in the direction of \( w = -2i + j + k \).

\[\text{<Sol>} (a) \quad \nabla f(x, y, z) = (\sin y z, x \cos y z \cdot z, x \cos y z \cdot y) \]

\[= (\sin(yz), xz \cos(yz), xy \cos(yz))\]

(b) \[\nabla f \bigg|_{(1, 3, 0)} = (\sin 0, 0, 3 \cos 0) = (0, 0, 3)\]

\[
(0, 0, 3) \cdot \frac{(-2, 1, 1)}{\sqrt{4 + 1 + 1}} = \frac{1}{\sqrt{6}} \cdot 3 = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}.
\]
Ex.

1. \( f(x,y) = xe^y \). Find the rate of change of \( f \) at \( P(2,0) \) in the direction from \( P(2,0) \) to \( (\frac{3}{2}, 2) \).

\[ \nabla f(2,0) = (e^0, 2e^0) = (1, 2) \]

\[ 1u = (-\frac{3}{2}, 2) \Rightarrow \frac{14}{11u} = \frac{(\frac{3}{2}, 2)}{\sqrt{\frac{9}{4} + 4}} = \frac{2}{\sqrt{35}}(-\frac{3}{2}, 2) \]

\[ \Rightarrow \nabla f(2,0) \cdot \frac{14}{11u} = \frac{2}{\sqrt{35}}(-\frac{3}{2} + 4) = \frac{3}{\sqrt{35}} \cdot \frac{5}{2} = \frac{3}{2} \]

2. In what direction does \( f \) have the maximum rate of change? What is this maximum rate of change?

\[ |\nabla f \cdot u| = |\nabla f| \cdot |u| \cdot \cos \theta \]

\[ = |\nabla f| \cdot \cos \theta \neq 0 \]

So maximum of the rate of change \( \theta = 180^\circ \) or \( 0^\circ \).

So \( |\nabla f| = \sqrt{5} \).
Ex. Suppose that the temperature at \((x, y, z)\) in space is given by

\[
T(x, y, z) = \frac{80}{(1 + x^2 + 2y^2 + 3z^2)} \quad \text{(measured in Celsius)}
\]

In which direction does \(T\) increase fastest at \((1, 1, -2)\)?

What's the maximum rate of increase?

\[
\nabla T(x, y, z) = \left( \frac{-160x}{(1 + x^2 + 2y^2 + 3z^2)^2}, \frac{-320y}{(1 + x^2 + 2y^2 + 3z^2)^2}, \frac{-480z}{(1 + x^2 + 2y^2 + 3z^2)^2} \right)
\]

\[
\nabla T(1,1,-2) = \left( \frac{-160}{(1 + 1^2 + 2 \cdot 1^2 + 3(-2)^2)^2}, \frac{-320}{256}, \frac{960}{256} \right) = \left( \frac{-160}{256}, \frac{-320}{256}, \frac{960}{256} \right) = \left( \frac{-5}{8}, \frac{-16}{8}, \frac{90}{8} \right)
\]

\[
|\nabla T(1,1,-2)| = \frac{5}{8} \sqrt{1 + 4 + 36} = \frac{5}{8} \sqrt{41}
\]

in the direction of \((-\frac{5}{8}, \frac{-16}{8}, \frac{30}{8})\)

\[
= \frac{5}{8} \cdot (-1, -2, 6)
\]
Tangent Planes to Level Surfaces

$S$ is a surface with $F(x, y, z) = k$, level surface of a function $F$ of 3 variables. Let $P(x_0, y_0, z_0)$ be on $S$. Let $C$ be any curve that lies on $S$ and passes through $P$.

\[ F(x(t), y(t), z(t)) = k \]

\[ \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0 \]

\[ \iff \nabla F \cdot (x'(t), y'(t), z'(t)) = 0 \]

\[ \nabla F \cdot v(t) = 0 \]

i.e. $\nabla F$ & $v(t)$ are orthogonal perpendicular.

So, the tangent plane to the level surface $F(x, y, z) = k$ at $(x_0, y_0, z_0)$ is the plane that passes through $(x_0, y_0, z_0)$ and has normal vector $\nabla F(x_0, y_0, z_0)$. 
\[ F_x (x_0, y_0, z_0) (x-x_0) + F_y (x_0, y_0, z_0) (y-y_0) + F_z (x_0, y_0, z_0) (z-z_0) = 0 \]

Normal line to \( S \) at \( P(x_0, y_0, z_0) \) is

\[
\frac{x-x_0}{F_x (x_0, y_0, z_0)} = \frac{y-y_0}{F_y (x_0, y_0, z_0)} = \frac{z-z_0}{F_z (x_0, y_0, z_0)}
\]

Ex. Find the tangent plane at normal line at \((-2, 1, -3)\) to

\[ \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \]

\[ F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} \]

\[ F_x (2, 1, -3) = \left. \frac{x}{2} \right|_{(-2, 1, -3)} = -1 \]

\[ F_y (-2, 1, -3) = \left. 2y \right|_{(-2, 1, -3)} = 2 \]

\[ F_z (-2, 1, -3) = \left. \frac{z}{3} \right|_{(-2, 1, -3)} = -\frac{2}{3} \]

\[ S_0: -(x+2) + 2(y-1) - \frac{2}{3} (z+3) = 0 \]

\[ 3x - 6y + 2z + 18 = 0 \]

Normal line:

\[
\frac{x+2}{-1} = -\frac{y-1}{2} = \frac{z+3}{-\frac{2}{3}}
\]
The gradient vector $\nabla f(x_0, y_0, z_0)$

(1) gives the direction of fastest increase of $f$.

(2) is orthogonal to the level surface $S$ of $f$. 