BIFURCATION AND OF THE GENERALIZED COMPLEX
GINZBURG–LANDAU EQUATION

JUNGHO PARK

Abstract. We study in this paper the bifurcation and stability of the solutions of the complex Ginzburg–Landau equation (CGLE). We investigate two different modes of CGLE. We study the first mode of CGLE which has only cubic unstable nonlinear term and later we also study the second mode of CGLE which has both cubic and quintic nonlinear terms. The solutions considered in cubic CGLE bifurcate from the trivial state supercritically in some parameter range. However, for the cubic-quintic CGLE, solutions bifurcate from the trivial state subcritically. Due to the global attractor, we achieved a saddle node bifurcation point $\lambda_c$. We also study the steady state bifurcation of the CGLE and verify that the bifurcated invariant sets $\Sigma_k$ contain steady state solutions.

1. Introduction

The complex Ginzburg–Landau equation (CGLE) is a widely studied partial differential equation with applications in many areas of science. If the dissipative terms in the CGLE are neglected, the equation is reduced to the nonlinear Schrödinger equation. It has become a model problem for the study of nonlinear evolution equations with chaotic spatio-temporal dynamics. In this paper we work with the CGLE which reads

\begin{equation}
\frac{\partial u}{\partial t} = \rho u + (\epsilon_0 + i\rho_0)\Delta u + (\epsilon_1 + i\rho_1)|u|^2u - (\epsilon_2 + i\rho_2)|u|^4u,
\end{equation}

where the unknown function $u : \Omega \times [0, \infty) \to \mathbb{C}$ is a complex-valued function and $\Omega \subset \mathbb{R}^n$ is an open, bounded, and smooth domain in $\mathbb{R}^n$ ($1 \leq n \leq 3$). The parameters $\epsilon_i$ and $\rho_i (i = 0, 1, 2)$ are real numbers and $\rho$ is the system parameter. This is the equation we shall consider, supplemented with either the Dirichlet boundary condition,

\begin{equation}
u|_{\partial \Omega} = 0,
\end{equation}

or the space periodic boundary condition,

\begin{equation}
\Omega = (0, 2\pi)^n\text{ and } u\text{ is }\Omega-\text{periodic},
\end{equation}

1991 Mathematics Subject Classification. 35, 37.

Key words and phrases. generalized Ginzburg–Landau equation, subcritical and supercritical bifurcation, saddle node, stability.

The work was supported in part by the Office of Naval Research, and by the National Science Foundation.
and suitable initial data

\begin{equation}
\tag{1.4}
\begin{aligned}
&u(x, 0) = \phi + i\psi.
\end{aligned}
\end{equation}

This equation is a canonical model for weakly-nonlinear, dissipative systems and for this reason, it arises in a variety of settings, including nonlinear optics, fluid dynamics, chemical physics, mathematical biology, condensed matter physics, and statistical mechanics. In fluid dynamics the CGLE is found, for example, in the study of Poiseuille flow, the nonlinear growth of convection rolls in the Rayleigh–Bénard problem and Taylor–Couette flow. In this case, the bifurcation parameter \( \rho \) plays the role of a Reynolds number. The equation also arises in the study of chemical systems governed by reaction-diffusion equations. The CGLE plays the role of a simplified set of fluid dynamic equations in the following sense. The Navier-Stokes equations can be characterized as an infinite dimensional dynamical system whose behavior is often dominated by long wavelength instabilities with short wavelength dissipation (due to viscosity) and a nonlinear mode coupling (the convection term) that provides an energy transport from the long to the short wavelengths. A useful bifurcation parameter for the Navier-Stokes equations, the Reynolds number, is the ratio of the product of a long wavelength velocity and length scale and the short wavelength dissipation. Quiescent low Reynolds number flows are characterized by a low ratio of driving to damping, while the often chaotic or turbulent high Reynolds number flows correspond to a high ratio of them. The same qualitative features are present in the CGLE.

There are extensive studies for the CGLE and we refer, in particular, to [1, 2, 4, 5, 6, 13, 15] and the references therein for studies related to the global attractors, inertial manifolds, soft and hard turbulences, coherent structure, kink and solitons, stabilities and bifurcation theories described by the CGLE. If we take a look at the equation (1.1), then we can realize that we have choices to set any three coefficients \( \varepsilon_i \) and \( \rho_i \) [15]. When \( \varepsilon_1 < 0 \), the equation has a subcritical bifurcation at some critical values, and one often takes \( \varepsilon_2 = \rho_2 = 0 \) and \( \varepsilon_0 = -\varepsilon_1 = 1 \) since \( \varepsilon_2 \) is not necessary for stability (We will briefly mention about it at the end of the paper). Therefore, the equation has two parameters \( \rho_0 \) and \( \rho_1 \). When \( \varepsilon_1 > 0 \), the equation has a supercritical bifurcation, and we must retain \( \varepsilon_2 > 0 \) which allows us to avoid an expected blowup of the solutions. It is usual to take \( \varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1 \) and the equation has the three parameters \( \rho_0, \rho_1 \) and \( \rho_2 \).

The first supercritical case was recently studied in [13] and had supercritical bifurcations as \( \rho \) crosses some critical critical numbers, for both Dirichlet and periodic boundary conditions. In this paper we study the subcritical case i.e., cubic nonlinear term has unstable real term and the results can be summarized as follows.
First, we consider CGLE in the case of \( \varepsilon_1 = 1 > 0 \) and \( \varepsilon_2 = \rho_2 = 0 \) (so called cubic CGLE) with (1.3):

\[
(1.5) \quad \frac{\partial u}{\partial t} = \rho u + (1 + i\rho_0)\Delta u + (1 + i\rho_1)|u|^2u,
\]

supplemented with a parameter range \((\rho_0, \rho_1)\). If \( \rho_0 \) and \( \rho_1 \) stay in the region of \( 1 - \frac{\rho_1}{\rho_0} < 0 \), the solutions of (1.5) bifurcate from \( u = 0 \) “supercritically” in spite of the fact that the sign of the real part of the nonlinear term signals a subcritical bifurcation. See Figure 1.1.

![Figure 1.1. Supercritical bifurcation diagram for cubic CGLE. Solid lines denote stable states and dashed lines unstable states. For \( \rho > 0 \), the problem bifurcates supercritically.](image)

Second, we consider CGLE in the case of nonzero cubic and quintic terms \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \) (so called cubic-quintic CGLE) with either (1.2) or (1.3):

\[
(1.6) \quad \frac{\partial u}{\partial t} = \rho u + (1 + i\rho_0)\Delta u + (1 + i\rho_1)|u|^2u - (1 + i\rho_2)|u|^4u.
\]

If we let \( \lambda_1 \) be the first eigenvalue of \(-\Delta\) under (1.2), then the equation bifurcates subcritically from \( u = 0 \) for \( \rho < \lambda_1 \). Moreover, by the help of the global attractor near \( \rho = 0 \), we obtain a saddle node bifurcation point \((u, \rho) = (u^*, \lambda_c)\) \((0 < \lambda_c < \lambda_1)\) from which two branches of solutions \( \Sigma_1 \) and \( \Sigma_2 \) bifurcate. We can easily see that \( \Sigma_1 \) is a repeller and convergent to \((0, \lambda_1)\) as \( \rho \to \lambda_1 - \) and \( \Sigma_2 \) is an attractor and extends to \( \rho > \lambda_1 \). We can also prove that these two bifurcated invariant sets are \( S^1 \). Considering the
periodic boundary condition (1.3), we obtain the same result as the Dirichlet boundary case, for $\rho < 0$. See Figure 1.2.

It is worth mentioning that the above two results can be verified using a new notion of bifurcation theory which was developed in [14] and the results coincide with those in [1, 4, 6, 15].

The paper is organized as follows. In section 2, we recall the center manifold theory as a reduction method and attractor bifurcation theory. In section 3, we study cubic CGLE followed by cubic-quintic CGLE in section 4 and 5. We shall describe the global attractor of CGLE and give concluding remarks in section 6.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{bifurcation_diagram.png}
\caption{Subcritical bifurcation diagram for cubic-quintic CGLE. Solid lines denote stable states and dashed lines unstable states. For $\lambda_c < \rho < \lambda_1$, $\Sigma_1$ represents the inner circle $(e^{i\theta}u^0_1, \rho)$ and $\Sigma_2$ represents the outer circle $(e^{i\theta}u^0_2, \rho)$, both of which are bifurcated from nonzero point $(u^*, \lambda_c) = (e^{i\theta}u^*)$.}
\end{figure}

2. Center Manifold and Attractor Bifurcation Theory

2.1. Center manifold approximation. The purpose of this section is to recall some of the results of center manifold theorem, which is a powerful tool for the reduction method and for dynamic bifurcation of abstract nonlinear evolution equations developed in [14].
Let $H$ and $H_1$ be two Hilbert spaces, and $H_1 \hookrightarrow H$ be a dense and compact inclusion. Consider the following nonlinear evolution equations

\begin{align}
(2.1) & \quad \frac{du}{dt} = L_\lambda u + G(u, \lambda), \\
(2.2) & \quad u(0) = u_0,
\end{align}

where $u : [0, \infty) \to H$ is the unknown function, $\lambda \in \mathbb{R}$ is the system parameter, and $L_\lambda : H_1 \to H$ are parameterized linear completely continuous fields continuously depending on $\lambda \in \mathbb{R}$, which satisfy

\begin{align}
(2.3) & \quad \begin{cases}
L_\lambda = -A + B_\lambda & \text{a sectorial operator,} \\
A : H_1 \to H & \text{a linear homeomorphism,} \\
B_\lambda : H_1 \to H & \text{the parameterized linear compact operators.}
\end{cases}
\end{align}

We can see that $L_\lambda$ generates an analytic semi-group $\{e^{-tL_\lambda}\}_{t \geq 0}$ and we can define fractional power operators $L_\alpha^\lambda$ for any $0 \leq \alpha < 1$ with domain $H_\alpha^\lambda = D(L_\alpha^\lambda)$ such that $H_{\alpha_1} \subset H_{\alpha_2}$ if $\alpha_2 < \alpha_1$, and $H_0 = H$.

We now assume that the nonlinear terms $G(\cdot, \lambda) : H_\alpha^\lambda \to H$ for some $0 \leq \alpha < 1$ are a family of parameterized $C^r$ bounded operators ($r \geq 1$) continuously depending on the parameter $\lambda \in \mathbb{R}$, such that

\begin{align}
(2.4) & \quad G(u, \lambda) = o(\|u\|_{H_\alpha}), \quad \forall \lambda \in \mathbb{R}.
\end{align}

For the linear operator $A$ we assume that there exists a real eigenvalue sequence $\{\rho_k\} \subset \mathbb{R}$ and an eigenvector sequence $\{e_k\} \subset H_1$, i.e.,

\begin{align}
(2.5) & \quad \begin{cases}

Ae_k = \rho_k e_k, \quad e_k = x_k + iy_k \\
Re\rho_k \to \infty \text{ as } k \to \infty, \\
|Im\rho_k/(Re\rho_k + a)| \leq C \text{ for some constants } a, C > 0
\end{cases}
\end{align}

where $\{x_k, y_k\}$ is an orthogonal basis of $H$.

For the compact operator $B_\lambda : H_1 \to H$, we also assume that there is a constant $0 < \theta < 1$ such that

\begin{align}
(2.6) & \quad B_\lambda : H_\theta \to H \text{ bounded}, \quad \forall \lambda \in \mathbb{R}.
\end{align}

We know that the operator $L_\lambda = -A + B_\lambda$ satisfying (2.5) and (2.6) is a sectorial operator. It generates an analytic semigroup $\{S_\lambda(t)\}_{t \geq 0}$. Then the solution of (2.1) and (2.2) can be expressed as

\begin{align}
(2.7) & \quad u(t, u_0) = S_\lambda(t)u_0, \quad t \geq 0.
\end{align}

We assume that the spaces $H_1$ and $H$ can be decomposed into

\begin{align}
H_1 = E_1^\lambda \oplus E_2^\lambda, \quad \dim E_1^\lambda < \infty, \quad \text{near } \lambda_0 \in \mathbb{R}^1, \\
H = \bar{E}_1^\lambda \oplus \bar{E}_2^\lambda, \quad \bar{E}_1^\lambda = E_1^\lambda, \quad \bar{E}_2^\lambda = \text{closure of } E_2^\lambda \text{ in } H,
\end{align}
where \( E^1_\lambda \) and \( E^2_\lambda \) are two invariant subspaces of \( L_\lambda \), i.e., \( L_\lambda \) can be decomposed into \( L_\lambda = L^1_\lambda \oplus L^2_\lambda \) such that for any \( \lambda \) near \( \lambda_0 \),

\[
\begin{cases}
L^1_\lambda = L_\lambda|_{E^1_\lambda} : E^1_\lambda \to \tilde{E}^1_\lambda, \\
L^2_\lambda = L_\lambda|_{E^2_\lambda} : E^2_\lambda \to \tilde{E}^2_\lambda,
\end{cases}
\]

where all eigenvalues of \( L^2_\lambda \) possess negative real parts, and all eigenvalues of \( L^1_\lambda \) possess nonnegative real parts at \( \lambda = \lambda_0 \).

Thus, for \( \lambda \) near \( \lambda_0 \), \( (2.1) \) can be rewritten as

\[
\begin{cases}
\frac{dx}{dt} = L^1_\lambda x + G_1(x, y, \lambda), \\
\frac{dy}{dt} = L^2_\lambda y + G_2(x, y, \lambda),
\end{cases}
\]

where \( u = x + y \in H_1 \), \( x \in E^1_\lambda \), \( y \in E^2_\lambda \), \( G_i(x, y, \lambda) = P_i G(u, \lambda) \), and \( P_i : H \to \tilde{E}_i \) are canonical projections. Furthermore, we let

\[ E^2_\lambda(\alpha) = E^2_\lambda \cap H_\alpha, \]

with \( \alpha \) given by \( (2.4) \).

**Theorem 2.1. (Center Manifold Theorem)** Assume \( (2.4)–(2.8) \). Then there exists a neighborhood of \( \lambda_0 \) given by \( |\lambda - \lambda_0| < \delta \) for some \( \delta > 0 \), a neighborhood \( B_\lambda \subset E^1_\lambda \) of \( x = 0 \), and a \( C^1 \) function \( \Phi(\cdot, \lambda) : B_\lambda \to E^2_\lambda(\alpha) \), depending continuously on \( \lambda \), such that

1. \( \Phi(0, \lambda) = 0 \), \( D_x \Phi(0, \lambda) = 0 \).
2. The set

\[ M_\lambda = \left\{ (x, y) \in H_1 \mid x \in B_\lambda, y = \Phi(x, \lambda) \in E^2_\lambda(\alpha) \right\}, \]

called center manifold, is locally invariant for \( (2.1) \), i.e., for any \( u_0 \in M_\lambda \),

\[ u_\lambda(t, u_0) \in M_\lambda, \quad \forall \ 0 \leq t < t_{u_0}, \]

for some \( t_{u_0} > 0 \), where \( u_\lambda(t, u_0) \) is the solution of \( (2.1) \).
3. If \( (x_\lambda(t), y_\lambda(t)) \) is a solution of \( (2.9) \), then there are \( \beta_\lambda > 0 \) and \( k_\lambda > 0 \) with \( k_\lambda \) depending on \( (x_\lambda(0), y_\lambda(0)) \) such that

\[ \|y_\lambda(t) - \Phi(x_\lambda(t), \lambda)\|_H \leq k_\lambda e^{-\beta_\lambda t}. \]

By the help of Center Manifold Theorem, we obtain the following bifurcation equation reduced to the finite dimensional system

\[ \frac{dx}{dt} = L^1_\lambda x + G_1(x + \Phi(x), \lambda), \]

for \( x \in B_\lambda \subset E^1_\lambda \).

Now we recall an approximation of the center manifold function which will be used in the proof of Theorem 4.1; see [14] for details. Let the nonlinear operator \( G \) be given by

\[
G(u, \lambda) = G_k(u, \lambda) + o(|x|^k),
\]

\( (2.10) \)
where $G_k(u, \lambda)$ is a $k$-multilinear operator ($k \geq 2$).

**Theorem 2.2.** (T. Ma and S. Wang, [14]) Under the conditions in Theorem 2.1 and (2.10), we have the following center manifold function approximation:

$$
\Phi(x, \lambda) = (-L_\lambda^2)^{-1}P_2G_k(x, \lambda) + O(|\text{Re}\beta(\lambda)| \cdot ||x||^5) + o(||x||^6),
$$

where $\beta(\lambda) = (\beta_1(\lambda), \cdots, \beta_m(\lambda))$ are the eigenvalues of $L_\lambda^1$.

### 2.2. Attractor bifurcation theory.

In this subsection, we shall recall abstract attractor bifurcation theory [14]. Moreover, we shall provide a sufficient condition which implies that the bifurcated attractor of the system (2.1) is homeomorphic to $S^1$.

**Definition 2.3.** A set $\Sigma \subset H$ is called an invariant set of (2.1) if $S(t, \lambda)\Sigma = \Sigma$ for any $t \geq 0$. An invariant set $\Sigma_\lambda \subset H$ of (2.1) is called an attractor if $\Sigma$ is compact, and there exists a neighborhood $U \subset H$ of $\Sigma$ such that for any $\varphi \in U$ we have

$$
\lim_{t \to \infty} \text{dist}_H(u(t, \varphi), \Sigma) = 0.
$$

The largest open set $U$ satisfying the above convergence is called the basin of attraction of $\Sigma$.

**Definition 2.4.**

1. We say that (2.1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ an invariant set $\Sigma_\lambda$ if there exists a sequence of invariant sets $\{\Sigma_\lambda_n\}_{n \in \mathbb{N}}$ of (2.1), $0 \notin \Sigma_\lambda_n$, such that

$$
\lim_{n \to \infty} \lambda_n = \lambda_0,
$$

$$
\lim_{n \to \infty} \max_{x \in \Sigma_\lambda_n} |x| = 0.
$$

2. If the invariant sets $\Sigma_\lambda$ are attractors of (2.1), then the bifurcation is called attractor bifurcation.

3. If $\Sigma_\lambda$ are attractors and are homotopy equivalent to an $m$-dimensional sphere $S^m$, then the bifurcation is called an $S^m$-attractor bifurcation.

Let the eigenvalues (counting multiplicity) of $L_\lambda$ be given by $\beta_k(\rho) \in \mathbb{C}$ ($k \geq 1$).

Suppose that

$$
\text{Re}\beta_i(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0 \\ > 0 & \text{if } \lambda > \lambda_0, \end{cases} \quad (1 \leq i \leq m),
$$

$$
\text{Re}\beta_j(\lambda_0) < 0 \quad (m + 1 \leq j).
$$

Let the eigenspace of $L_\lambda$ at $\lambda_0$ be

$$
E_0 = \bigcup_{i=1}^{m} \left\{ u \in H_1 \mid (L_{\lambda_0} - \beta_i(\lambda_0))^k u = 0, k = 1, 2, \cdots \right\}.
$$
The following dynamic bifurcation theorem for (2.1) is the key idea to studying CGLE in this paper.

**Theorem 2.5.** (Attractor Bifurcation) Assume that (2.3)–(2.6) hold and $u = 0$ is a locally asymptotically stable equilibrium point of (2.1) at $\lambda = \lambda_0$. Then the following assertions hold.

1. The equation (2.1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ an attractor $A_{\lambda}$ for $\lambda > \lambda_0$, with $m - 1 \leq \dim A_{\lambda} \leq m$, which is connected if $m > 1$.
2. The attractor $A_{\lambda}$ is a limit of a sequence of $m$-dimensional annulus $M_k$ with $M_{k+1} \subset M_k$; in particular if $A_{\lambda}$ is a finite simplicial complex, then $A_{\lambda}$ has the homotopy type of $S^{m-1}$.
3. For any $u_{\lambda} \in A_{\lambda}$, $u_{\lambda}$ can be expressed as $u_{\lambda} = v_{\lambda} + o(\|v_{\lambda}\|_{H^1})$, $v_{\lambda} \in E_0$.
4. If $u = 0$ is globally stable for (2.1) at $\lambda = \lambda_0$, then for any bounded open set $U \subset H$ with $0 \in U$ there is an $\varepsilon > 0$ such that as $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, the attractor $A_{\lambda}$ bifurcated from $(0, \lambda_0)$ attracts $U/\Gamma$ in $H$, where $\Gamma$ is the stable manifold of $u = 0$ with codimension $m$.

So far we introduced crucial definitions and theorems which established the existence of bifurcation. We now introduce another theorem related to the structure of bifurcated solutions.

Let $v$ be a two-dimensional $C^r (r \geq 1)$ vector field given by
\begin{equation}
(2.13) \quad v_{\lambda}(x) = \lambda x - G(x, \lambda),
\end{equation}
for $x \in \mathbb{R}^2$. Here, $G(x, \lambda)$ is defined as in (2.10) and satisfies an inequality
\begin{equation}
(2.14) \quad C_1 |x|^{k+1} \leq < G_k(x, \lambda), x >_H \leq C_2 |x|^{k+1},
\end{equation}
for some constants $0 < C_1 < C_2$ and $k = 2m + 1$, $m \geq 1$.

**Theorem 2.6.** (T. Ma and S. Wang) Under the condition (2.14), the vector field (2.13) bifurcates from $(x, \lambda) = (0, \lambda_0)$ to an attractor $\Sigma_{\lambda}$ for $\lambda > \lambda_0$, which is homeomorphic to $S^1$. Moreover, one and only one of the following is true.

1. $\Sigma_{\lambda}$ is a periodic orbit,
2. $\Sigma_{\lambda}$ consists of infinitely many singular points, or
3. $\Sigma_{\lambda}$ contains at most $2(k + 1) = 4(m + 1)$ singular points, and has $4N + n(N + n \geq 1)$ singular points, $2N$ of which are saddle points, $2N$ of which are stable node points (possibly degenerate), and $n$ of which have index zero.

**2.3. Steady state bifurcation theory.** Consider a parameter family of nonlinear operator equations
\begin{equation}
(2.15) \quad L_{\lambda} u + G(u, \lambda) = 0,
\end{equation}
where $L_{\lambda} : H_1 \to H$ is a completely continuous field and $G(\cdot, \lambda) : H_1 \to H$ is a $C^\infty$ operator satisfying (2.10).
Let \( \{e_1, \cdots, e_r\} \) and \( \{e_1^*, \cdots, e_r^*\} \in H_1 \) be the eigenvectors of \( L_\lambda \) and \( L_\lambda^* \) at \( \lambda = \lambda_0 \), respectively. Here \( r \leq m \) is the geometric multiplicity of \( \beta_1(\lambda_0) \). Let
\[
a^i_{j_1 \cdots j_k} = < G_k(e_{j_1}, \cdots, e_{j_k}, \lambda_0), e_i^* >_{H},
\]
and
\[
\Sigma = \{ (u, \lambda) \in H_1 \times \mathbb{R} \mid L_\lambda u + G(u, \lambda) = 0, \; u \neq 0, \; \lambda < \lambda_0 \}.
\]
The steady state solution \( u = 0 \) of (2.15) is called \( k \)-th order nondegenerate at \( \lambda = \lambda_0 \) if \( x = (x_1, \cdots, x_r) = 0 \) is an isolated singular point of
\[
\sum_{j_1, \cdots, j_k=1}^r a^i_{j_1 \cdots j_k} x_{j_1} \cdots x_{j_k} = 0, \; (1 \leq i \leq r).
\]

**Theorem 2.7.** Let \( \Sigma_0 \subset \Sigma \) be the connected component of \( \Sigma \) containing \((0, \lambda_0)\). If \( u = 0 \) is a \( k \)-th order nondegenerate at \( \lambda = \lambda_0 \), then one of the following assertions is true.

1. \( \Sigma_0 \) is unbounded
2. \( \Sigma_0 \) contains \((0, \lambda_1)\) with \( \lambda_1 < \lambda_0 \) such that there are some eigenvalues \( \beta_j(\lambda) \) of \( L_\lambda \) at \( \lambda = \lambda_1 \) with \( \beta_j(\lambda_1) = 0 \) or
3. There exists a point \((v_0, \nu) \in H_1 \times \mathbb{R}\) with \( \nu < \lambda_0 \) such that
\[
\Sigma_0 \cap (H_1 \times \{\lambda\}) = \begin{cases} \emptyset & \text{if } \lambda < \nu, \\ (v_0, \nu) & \text{if } \lambda = \nu \\ \Gamma_1(\lambda) + \Gamma_2(\lambda) & \text{if } \nu < \lambda \leq \lambda_0, \end{cases}
\]
where \( \Gamma_1(\lambda_0) = (0, \lambda_0), \Gamma_2(\lambda_0) \neq (0, \lambda_0), \Gamma_1(\lambda) \neq \emptyset \) and \( \Gamma_2(\lambda) \neq \emptyset \).

3. **Bifurcation of Cubic CGLE with Periodic Boundary Condition**

In this section, we study cubic CGLE under the periodic boundary condition. We need the following function spaces:
\[
H = L^2_{\text{per}}(\Omega) = \{ u \in L^2(\Omega) \mid u \text{ is } \Omega - \text{periodic} \},
\]
\[
H_1 = H^2_{\text{per}}(\Omega) = \{ u \in H^2(\Omega) \mid u \text{ is } \Omega - \text{periodic} \}.
\]
We now have the following supercritical bifurcation theorem for cubic CGLE.

**Theorem 3.1.** Let two coefficients \( \rho_0 \) and \( \rho_1 \) satisfy
\[
1 - \frac{\rho_1}{\rho_0} < 0,
\]
then the following assertions hold.

1. \( u = 0 \) is globally asymptotically stable for \( \rho \leq 0 \).
2. The problem (1.5) bifurcates from \((u, \rho) = (0, 0)\) to an attractor \( \Sigma_\rho \) as \( \rho \) crosses 0.
Proof. Let \( u = u_1 + iu_2 \) then (1.5) can be equivalently written as

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \Delta u_1 - \rho_0 \Delta u_2 + \rho u_1 + |u|^2 u_1 - \rho_1 |u|^2 u_2, \\
\frac{\partial u_2}{\partial t} &= \rho_0 \Delta u_1 + \Delta u_2 + \rho u_2 + |u|^2 u_2 + \rho_1 |u|^2 u_1, \\
u_1(x, 0) &= \phi(x), \quad u_2(x, 0) = \psi(x).
\end{align*}
\]

(3.1)

The mappings \( L_\rho = -A + B_\rho \) and \( G : H_1 \rightarrow H \) are defined as

\[
\begin{align*}
-Au &= \begin{pmatrix}
\Delta u_1 - \rho_0 \Delta u_2 \\
\rho_0 \Delta u_1 + \Delta u_2
\end{pmatrix}, \\
B_\rho u &= \rho \begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}, \\
Gu &= \begin{pmatrix}
|u|^2 u_1 - \rho_1 |u|^2 u_2 \\
|u|^2 u_2 + \rho_1 |u|^2 u_1
\end{pmatrix}.
\end{align*}
\]

It is known that \( H_{1/2} = H_{1, \text{per}}^1 \), therefore \( G : H_{1/2} \rightarrow H \) is \( C^\infty \).

We know that the eigenvalue problem

\[
\begin{align*}
-\Delta e_k &= \lambda_k e_k, \\
e_k(x + 2j\pi) &= e_k(x)
\end{align*}
\]

(3.2)

has eigenvalues given by

\( \lambda_k = |k|^2 = k_1^2 + \cdots + k_n^2, \) \( k = (k_1, \cdots, k_n) \) and \( k_i = 0, 1, \cdots, (1 \leq i \leq n) \)

and the eigenvectors corresponding to \( \lambda_k \) are given by

\[
\sin(k_1 x_1 + \cdots + k_n x_n) \quad \text{and} \quad \cos(k_1 x_1 + \cdots + k_n x_n)
\]

It is easy to see that the eigenvalues of \( L_\rho = -A + B_\rho \) are given by

\( \beta_k = (\rho - |k|^2) + i|k|^2 \rho_0 \)

and the eigenvectors are

\[
\begin{align*}
\cos{kx} \pm i \cos{kx}, & \quad \cos{kx} \pm i \sin{kx}, \\
\sin{kx} \pm i \cos{kx}, & \quad \sin{kx} \pm i \sin{kx},
\end{align*}
\]

where \( kx = k_1 x_1 + \cdots + k_n x_n \).

For the first eigenvalue \( \beta_0 = \rho \) of the operator \( L_\rho : H_1 \rightarrow H \), we can see that it satisfies (2.11) and (2.12) for \( \lambda_0 = 0 \) and \( \beta_0 \) has multiplicity two.

From equation (3.1), we have

\[
\frac{1}{2} \frac{d}{dt} |u|^2_{L^2} = \rho |u|^2_{L^2} - |\nabla u|^2_{L^2} + |u|^4_{L^4}
\]

(3.3)

and

\[
\rho_0 |\nabla u|^2_{L^2} = \rho_1 |u|^4_{L^4}.
\]

(3.4)

Replacing (3.3) by (3.4), we have

\[
\frac{1}{2} \frac{d}{dt} |u|^2_{L^2} = \rho |u|^2_{L^2} + (1 - \frac{\rho_1}{\rho_0}) |u|^4_{L^4}.
\]

(3.5)
Since \(1 - \frac{\rho_1}{\rho_0} < 0\), we have a following inequality
\[
\frac{1}{2} \frac{d}{dt} |u|^2_{L^2} \leq \rho |u|^2_{L^2} + (1 - \frac{\rho_1}{\rho_0}) |\Omega|^{-1} |u|^4_{L^2}.
\]
Therefore, \(u = 0\) is globally asymptotically stable for \(\rho \leq 0\). Thanks to Theorem 2.5 we can conclude that the problem (1.5) has a supercritical bifurcation from the trivial solution.

The proof is complete. \(\square\)

4. Bifurcation of Cubic-Quintic CGLE with Dirichlet Boundary Condition

In the previous section we achieved a supercritical bifurcation in some parameter range. Outside of this region we need, at the least, quintic terms to saturate the explosive instability provided by the nonlinear cubic term. In this section, we explore attractor bifurcation of the cubic-quintic CGLE under the Dirichlet boundary condition.

Let \(\lambda_1\) be the first eigenvalue of \(-\Delta\) with the Dirichlet boundary condition (1.2). Then we have the following bifurcation theorem for the cubic-quintic CGLE.

**Theorem 4.1.** For the Dirichlet boundary condition we have the following assertions

1. \(u = 0\) is locally asymptotically stable for \(\rho < \lambda_1\) and unstable for \(\rho \geq \lambda_1\).
2. The problem (1.6) bifurcates from \((u, \rho) = (0, \lambda_1)\) to an invariant set \(\Sigma_0\) and has no bifurcation on \(\rho > \lambda_1\).
3. There exists a saddle node bifurcation point \(0 < \lambda_c < \lambda_1\) of the problem (1.6).
4. At \(\rho = \lambda_c\), there is an invariant set \(\Sigma_0 = \Sigma_{\lambda_c}\) with \(0 \notin \Sigma_0\).
5. For \(\rho < \lambda_c\), there is no invariant set near \(\Sigma_0\).
6. For \(\lambda_c < \rho < \lambda_1\), there are two branches of invariant sets \(\Sigma_1\) and \(\Sigma_2\) and \(\Sigma_2\) extends to \(\rho \geq \lambda_1\) and near \(\lambda_1\) as well. Moreover, we have
   a. \(\Sigma_1\) is a repeller and \(\Sigma_2\) is an attractor with \(0 \notin \Sigma_2\).
   b. \(\Sigma_1\) is a cycle consisting of steady states \((i = 1, 2)\). In particular, the bifurcation is a Hopf bifurcation provided \(\rho_0 \neq 0\).

**Proof.** We divide the proof into several steps.

**Step 1.** Let \(H_1 = H^2(\Omega) \cap H_0^1(\Omega), H = L^2(\Omega)\) and \(u = u_1 + iu_2\). Then (1.6) with (1.4) can be equivalently written as

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \Delta u_1 - \rho_0 \Delta u_2 + \rho u_1 + |u|^2 u_1 - \rho_1 |u|^2 u_2 - |u|^4 u_1 + \rho_2 |u|^4 u_2, \\
\frac{\partial u_2}{\partial t} &= \rho_0 \Delta u_1 + \Delta u_2 + \rho u_2 + |u|^2 u_2 + \rho_1 |u|^2 u_1 - |u|^4 u_2 - \rho_2 |u|^4 u_1, \\
u_1(x, 0) &= \phi(x), \quad u_2(x, 0) = \psi(x).
\end{align*}
\]
The mappings $L_\rho = -A + B_\rho$ and $G : H_1 \to H$ are defined as

\begin{align*}
-Au &= \begin{pmatrix} \Delta u_1 - \rho_0 \Delta u_2 \\ \rho_0 \Delta u_1 + \Delta u_2 \end{pmatrix}, \\
B_\rho u &= \rho \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\
Gu &= \begin{pmatrix} |u|^2 u_1 - \rho_1 |u|^2 u_2 - |u|^4 u_1 + \rho_2 |u|^4 u_2 \\ |u|^2 u_2 + \rho_1 |u|^2 u_1 - |u|^4 u_2 - \rho_2 |u|^4 u_1 \end{pmatrix}.
\end{align*}

It is known that $H_{1/2} = H_0^1(\Omega)$, therefore $G : H_{1/2} \to H$ is $C^\infty$.

Let \{\lambda_k\} $\subset \mathbb{R}$ and \{\epsilon_k\} $\subset H^2(\Omega) \cap H_0^1(\Omega)$ be the eigenvalues and eigenvectors of $-\Delta$ with the Dirichlet boundary condition (1.2)

\begin{align*}
-\Delta e_k &= \lambda_k e_k, \\
e_k|_{\partial\Omega} &= 0.
\end{align*}

Note that

\[ 0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lambda_k \to \infty \text{ as } k \to \infty, \]

and \{\epsilon_k\} forms an orthogonal basis of $L^2(\Omega)$.

Since the eigenvalues of $A$ are given by

\[ \lambda_k \pm i\rho_0 \lambda_k, \quad k = 1, 2, \cdots \]

with the corresponding eigenvectors

\[ z_k = \epsilon_k + i\epsilon_k, \]

and \{\epsilon_k, i\epsilon_j | 1 \leq k, j < \infty\} is an orthogonal basis of $L^2(\Omega)$, the operator $L_\rho$ is a sectorial operator. Also note that the eigenvalues of $L_\rho = -A + B_\rho$ are given by

\begin{equation}
(\rho - \lambda_k) \pm i\rho_0 \lambda_k, \quad k = 1, 2, \cdots.
\end{equation}

From (4.2), we have the principles of the exchange of stabilities as follows:

\[ Re((\rho - \lambda_1) \pm i\rho_0 \lambda_1) \begin{cases} < 0 & \text{if } \rho < \lambda_1, \\
= 0 & \text{if } \rho = \lambda_1, \\
> 0 & \text{if } \rho > \lambda_1, \end{cases} \]

\[ Re((\rho - \lambda_k) \pm i\rho_0 \lambda_k) = \rho - \lambda_k < 0 \quad \text{for } \rho = \lambda_1 \quad (k \geq 2). \]

\textbf{Step 2.} We now find bifurcation equations of (1.6) reduced into the center manifold to find bifurcation points.
The space $H$ and $H_1$ can be decomposed into the form
\[ H_1 = E_1 \oplus E_2 \text{ and } H = E_1 \oplus \tilde{E}_2, \]
\[ E_1 = \{(x_1 + iy_1)e_1 \mid x_1, y_1 \in \mathbb{R}\}, \]
\[ E_2 = \left\{ \sum_{k=2}^{\infty} (x_k + iy_k)e_k \mid \sum_{k=2}^{\infty} \lambda_k^2(x_k^2 + y_k^2) < \infty \right\}, \]
\[ \tilde{E}_2 = \left\{ \sum_{k=2}^{\infty} (x_k + iy_k)e_k \mid \sum_{k=2}^{\infty} (x_k^2 + y_k^2) < \infty \right\}, \]
and the operator $L_\rho$ is decomposed into
\[
\left\{ \begin{array}{l}
L_\rho = L_1^\rho \oplus L_2^\rho, \\
L_1^\rho = L_\rho|_{E_1} : E_1 \to E_1, \quad L_2^\rho = L_\rho|_{E_2} : E_2 \to \tilde{E}_2.
\end{array} \right.
\]
It fulfills the conditions of Theorem 2.1. Therefore there exists a center manifold function $\Phi = \Phi_1 + i\Phi_2 : E_1 \to \tilde{E}_2$ satisfying
\[ \Phi(x_1, y_1) = o(|x_1|, |y_1|). \]
Let $u = v_1 + v_2 \in E_1 \oplus E_2$ then we have have the bifurcation equations of (1.6)
\[
\frac{dv_1}{dt} = L_1^\rho v_1 + G(v_1 + \Phi(v_1)),
\]
and it is equivalent to
\[
\begin{aligned}
\frac{dx_1}{dt} &= (\rho - \lambda_1)x_1 + \rho_0 \lambda_1 y_1 + P_1 G(v_1 + \Phi(v_1)), \\
\frac{dy_1}{dt} &= -\rho_0 \lambda_1 x_1 + (\rho - \lambda_1)y_1 + P_2 G(v_1 + \Phi(v_1)),
\end{aligned}
\]
where
\[ P_1 G(u) = \int_\Omega \left[ |u|^2 u_1 - \rho_1 |u|^2 u_2 - |u|^4 u_1 + \rho_2 |u|^4 u_2 \right] e_1 dx, \]
\[ P_1 G(u) = \int_\Omega \left[ |u|^2 u_2 + \rho_1 |u|^2 u_1 - |u|^4 u_2 - \rho_2 |u|^4 u_1 \right] e_1 dx. \]
Since
\[ u_1 = x_1 e_1 + \Phi_1(x_1, y_1), \quad u_2 = y_1 e_1 + \Phi_2(x_1, y_1) \]
and
\[ \Phi(x_1, y_1) = o(|x_1|, |y_1|), \]
we have
\[
P_1 G(x_1, y_1) = \int_\Omega \left[ (x_1^2 + y_1^2)x_1 - \rho_1 (x_1^2 + y_1^2) y_1 \right] e_1^1 dx
\]
\[ + \int_\Omega \left[ -(x_1^2 + y_1^2)^2 x_1 + \rho_2 (x_1^2 + y_1^2)^2 y_1 \right] e_1^0 dx
\]
\[ + o(|x_1|^3, |y_1|^3)
\]
\[ = \tau (x_1 - \rho_1 y_1) (x_1^2 + y_1^2) + o(|x_1|^3, |y_1|^3). \]
where $\tau = \int_{\Omega} e_1^4 dx > 0$. Therefore, (4.3) is rewritten as

\begin{align*}
\frac{dx_1}{dt} &= (\rho - \lambda_1)x_1 + \rho_0 \lambda_1 y_1 + \tau (x_1 - \rho_1 y_1) (x_1^2 + y_1^2) + o(|x_1|^3, |y_1|^3), \\
\frac{dy_1}{dt} &= -\rho_0 \lambda_1 x_1 + (\rho - \lambda_1) y_1 + \tau (y_1 + \rho_1 x_1) (x_1^2 + y_1^2) + o(|x_1|^3, |y_1|^3).
\end{align*}

Let

$$v(x_1, y_1) = -\tau \left( \frac{(x_1 - \rho_1 y_1)(x_1^2 + y_1^2)}{(y_1 + \rho_1 x_1)(x_1^2 + y_1^2)} \right).$$

Then, we have

$$<v(x_1, y_1), (x_1, y_1)> = -\tau (x_1^2 + y_1^2)^2 < 0
\text{ Step 3.}$$

Now, we consider the time-reversed semigroup $S_{\rho}(-t)$ generated by (4.4). We can see that $S_{\rho}(-t)$ has the same dynamic properties as the equations

\begin{align*}
\frac{dx_1}{dt} &= -(\rho - \lambda_1)x_1 - \rho_0 \lambda_1 y_1 - \tau (x_1 - \rho_1 y_1) (x_1^2 + y_1^2) + o(|x_1|^3, |y_1|^3), \\
\frac{dy_1}{dt} &= \rho_0 \lambda_1 x_1 - (\rho - \lambda_1) y_1 - \tau (y_1 + \rho_1 x_1) (x_1^2 + y_1^2) + o(|x_1|^3, |y_1|^3)
\end{align*}

and it is equivalent to

$$\frac{dz_1}{dt} = -(\rho - \lambda_1)z_1 + i\rho_0 \lambda_1 z_1 - \tau (1 + i\rho_1)|z_1|^2 z_1 + o(|z_1|^3).$$

Therefore we have

$$\frac{1}{2} \frac{d}{dt} |z_1|^2 = -(\rho - \lambda_1) |z_1|^2 - \tau |z_1|^4 + o(|z_1|^4).$$

Let $\widetilde{S}_{\rho}(t)$ be a semigroup which is generated by (4.6). Then from (4.7) we can see that $u = 0$ is the locally asymptotically stable solution of (4.6) for $\rho \geq \lambda_1$. Therefore, $u = 0$ is the locally asymptotically stable solution of (4.4) for $\rho < \lambda_1$ and unstable solution for $\rho \geq \lambda_1$. Hence, for the semigroup $S_{\rho}(t) = \widetilde{S}_{\rho}(-t)$ generated by (4.4) bifurcates from $(u, \rho) = (0, \lambda_1)$ to an attractor $\Sigma_k$ for $\rho < \lambda_1$ and it is obvious that it is a repeller for the time-reversed case.

If $\rho_0 \neq 0$, the bifurcation is the typical Hopf bifurcation since $\lambda_1$ is simple so that it $\Sigma_k = S^1$. Moreover, if $\rho = 0$ then we can infer (4.5) and Theorem 2.6 that bifurcated invariant set $\Sigma_k$ is homeomorphic to $S^1$.

**Step 4.** From equation (4.7) and the Hölder’s inequality, we have

$$\frac{1}{2} \frac{d}{dt} ||z_1||_{L^2}^2 \leq -(\rho - \lambda_1) ||z_1||_{L^2}^2 - \frac{\tau}{||\Omega||} ||z_1||_{L^2}^4$$

which implies that $u = 0$ is the globally asymptotically stable steady state solution of (4.1) near $\rho = 0$. By the existence of global attractor of (1.6) and arguments of Theorem 2.7(see [14], for details), we can see that there exists a $\lambda_c (0 < \lambda_c < \lambda_1)$, such that if $\rho < \lambda_c$, the equation has no nonzero singular
points and it generates, at least, a cycle $\Sigma_0$ of singular point at $\rho = \lambda_c$. Moreover, if $\rho > \lambda_c$, it bifurcates from $\Sigma_0$ to two cycles $\Sigma_1$ and $\Sigma_2$, which consist of singular points, such that $\Sigma_1$ is a repeller and is as described in Theorem 2.7 and $\Sigma_2$ is an attractor with $0 \notin \Sigma_2$ at $\rho = \lambda_1$. We can describe the structure of them in detail as follows:

For $\rho > \lambda_c$, the equations (4.1) bifurcate from $(u, \rho) = (u^*, \lambda_c)\ (u^* \neq 0)$ to two singular points $u_1^\rho = e^{i\theta}u^\rho_1$ and $u_2^\rho = e^{i\theta}u^\rho_2$ in $H$ satisfying

$$\lim_{\rho \to \lambda_c + 0} u_1^\rho = u^*, \quad i = 1, 2,$$

$$\lim_{\rho \to \lambda_1 - 0} u_1^\rho = 0,$$

$$u_2^\rho \neq 0 \text{ at } \rho = \lambda_1,$$

and $u_2^\rho$ are in an attractor $\Sigma_2$ for $\rho > \lambda_c$, as shown in Figure 1.1.

**Step 5.** Now we want to consider the steady state bifurcation. If $\rho_0 \neq 0$ then the eigenvalues (4.2) of $L_\rho$ at $\rho = \lambda_1$ are nonzero so that $L_\rho : H_1 \to H$ is a linear homeomorphism at $\rho = \lambda_1$, which means that (4.1) has no steady state bifurcation. Thus we assume $\rho_0 = 0$ which is a necessary condition for steady state bifurcation. Since the first eigenvalue of the linear operator $L_\rho = \rho u + (1 + \rho_0)\Delta u$ in the steady state equation

$$\rho u + (1 + i\rho_0)\Delta u + (1 + i\rho_1)|u|^2u - (1 + i\rho_2)|u|^4u = 0$$

is simple when we restrict our concerns only to real valued function space, classical Krasnoselski theorem leads us to the steady state bifurcation for $\lambda_c < \rho < \lambda_1$. Since cubic-quintic CGLE is invariant under the gauge transformation

$$\psi \to \psi e^{i\theta}, \quad \theta \in \mathbb{R}^1,$$

the set $S_\rho$ of steady state solutions of the cubic-quintic CGLE appears as a cycle $S^1$. Since $\Sigma_\rho \subset \Sigma_k = S^1$, the bifurcated attractor $\Sigma_k$ ($k = 1, 2$) consists of only steady states.

The proof is complete. $\square$

5. **Bifurcation of Cubic-Quintic CGLE with Periodic Boundary Condition**

For the cubic-quintic CGLE with periodic boundary condition, we need the same function space described in Section 3. The operators associated with cubic-quintic CGLE with periodic boundary condition are as defined in Section 4. Referring eigenvalue problem (3.2), we have the first eigenvalue $\beta_0 = 0$ of the operator $L_\rho$ at $\rho = 0$. Let $e_0$ be the eigenvector corresponding to $\beta_0$ and

$$E_1 = \{ z_0 e_0 \mid z_0 \in \mathbb{C} \} \text{ and } E_2 = E_1^\perp.$$

Let $u(= u_1 + iu_2) = v_1 + v_2 \in E_1 \oplus E_2$ then

$$u = z_0 e_0 + \sum_{k \geq 1} z_k e_k = (x_0 + iy_0)e_0 + \sum_{k \geq 1} (x_k + iy_k)e_k.$$
where \( \{ e_k | e_k \text{ are eigenvectors of } \beta_k, \ k = 1, 2, \cdots \} \). Let 
\[ x = x_0 e_0, \ y = y_0 e_0 \]
then by the center manifold theorem, there exists a center manifold function 
\[ \Phi = \Phi_1 + i \Phi_2 : E_1 \rightarrow \tilde{E}_2 \] satisfying 
\[ \Phi(x, y) = o(|x|, |y|) \]
and 
\[ u_1 = x + \Phi_1(x, y), \ u_2 = y + \Phi_2(x, y). \]
Then the bifurcation equations of (1.6) are given by 
\[ \frac{dv_1}{dt} = \mathcal{L}_1 v_1 + G(v_1 + \Phi(v_1)) \]
and it is equivalent to 
\[
\begin{cases}
\frac{dx_0}{dt} = \rho x_0 + P_1 G(v_1 + \Phi(v_1)), \\
\frac{dy_0}{dt} = \rho y_0 + P_2 G(v_1 + \Phi(v_1)),
\end{cases}
\]
where 
\[
P_1 G(u) = \int_{\Omega} \left[ |u|^2 u_1 - \rho_1 |u|^2 u_2 - |u|^4 u_1 + \rho_2 |u|^4 u_2 \right] e_0 dx,
\]
\[
P_1 G(u) = \int_{\Omega} \left[ |u|^2 u_2 + \rho_1 |u|^2 u_1 - |u|^4 u_2 - \rho_2 |u|^4 u_1 \right] e_0 dx.
\]
In the same fashion as in the proof of Theorem 4.1, we can obtain the following bifurcation theorem.

**Theorem 5.1.** For the periodic boundary condition, we have the following assertions.

1. \( u = 0 \) is locally asymptotically stable for \( \rho < 0 \) and unstable for \( \rho \geq 0 \).
2. The problem (1.6) bifurcates from \((u, \rho) = (0, 0)\) to an invariant set \( \Sigma_k \) and has no bifurcation on \( \rho > \lambda_1 \).
3. There exists a saddle node bifurcation point \( \lambda_c < 0 \) of the problem (1.6).
4. For \( \lambda_c < \rho < 0 \), there are two branches of invariant sets \( \Sigma_1 \) and \( \Sigma_2 \) and \( \Sigma_2 \) extends to \( \rho \geq \lambda_1 \) and near \( \lambda_1 \) as well. Moreover, we have 
   a) \( \Sigma_1 \) is a repellor and \( \Sigma_2 \) is an attractor with \( 0 \notin \Sigma_i \) for \( i = 1, 2 \).
   b) \( \Sigma_i = S^1 \) is a cycle consisting of steady states (for \( i = 1, 2 \)). In particular, the bifurcation is a Hopf bifurcation provided \( \rho_0 \neq 0 \).

**Remark 5.2.** It is known that the saddle node bifurcation point \( \lambda_c \) can be calculated analytically under the periodic boundary condition, and it is given as \( \lambda_c = -\frac{1}{4} \), see [6, 15] for details.
Remark 5.3. We can replace the stable quintic nonlinear term with an arbitrary odd degree stable term. In fact, the quintic term is the least degree term among them.

Remark 5.4. If we add two stable cubic and quintic terms, the results are exactly same as in [13]. Therefore, we only need to take nonlinear cubic and stable quintic terms.

6. Global Attractor of CGLE and Concluding Remarks

6.1. The existence of a global attractor of CGLE. In this subsection, we shall describe the existence of a global attractor of CGLE which was used in the proof of Theorem 4.1. We shall follow the same functional setting, boundary conditions and eigenvalues as defined in previous sections.

For the equation under consideration, we will prove the existence of absorbing sets in $L^2(\Omega)$ and $H^2(\Omega) \cap H^1_0(\Omega)$ under the Dirichlet boundary condition, and proving the existence of absorbing sets amounts to proving a priori estimates.

To prove the existence of an absorbing set in $L^2(\Omega)$, we multiply (1.6) by complex conjugate $u^*$ of $u$ and integrate over $\Omega$, then we obtain

$$\frac{d}{dt} |u|^2_{L^2} + 2|\nabla u|^2_{L^2} + \int_{\Omega} 2|u|^6 - 2|u|^4 - 2\rho |u|^2 \, dx = 0.$$ 

Since

$$\frac{1}{2}|u|^6 - C_1 \leq |u|^6 - |u|^4 - \rho |u|^2, \quad \text{for some } C_1 > 0,$$

we have

$$\frac{d}{dt} |u|^2_{L^2} + 2|\nabla u|^2_{L^2} + \int_{\Omega} |u|^6 \, dx \leq 2C_1 |\Omega|,$$

where $|\Omega|$ is the measure of $\Omega$. Due to the Poincaré inequality, we have

$$\frac{d}{dt} |u|^2_{L^2} + 2\lambda_1 |u|^2_{L^2} \leq C_2, \quad C_2 = 2C_1 |\Omega|.$$ 

Using the classical Gronwall lemma, we see that

$$|u(t)|^2_{L^2} \leq |u_0|^2_{L^2} \exp(-2\lambda_1 t) + \frac{C_2}{2\lambda_1} (1 - \exp(-2\lambda_1 t)).$$

Thus

$$\limsup_{t \to +\infty} |u(t)|_{L^2} \leq \eta_0, \quad \eta_0^2 = \frac{C_2}{2\lambda_1}.$$ 

There exists an absorbing set $B_0$ in $L^2(\Omega)$, namely, any ball of $L^2(\Omega)$ centered at 0 of radius $\eta_0' > \eta_0$. If $B$ is a bounded set of $L^2(\Omega)$, included in a ball $B(0, R)$ of $L^2(\Omega)$, centered at 0 of radius $R$, then $S(t)B \subset B(0, \eta_0')$ for $t \geq t_0(B, \eta_0')$, where

$$t_0 = \frac{1}{2\lambda_1} \ln \frac{R^2}{(\eta_0')^2 - \eta_0^2}.$$
We now prove the existence of an absorbing set in $H^2(\Omega) \cap H^1_0(\Omega)$ and the uniform compactness of $S(t)$. Multiplying (1.6) by $-\Delta u^*$ and integrating over $\Omega$ again, we obtain

$$
- \int_{\Omega} \frac{\partial u}{\partial t} \Delta u^* dx = - (1 + i \rho_0) \int_{\Omega} |\Delta u|^2 dx - \rho \int_{\Omega} u \Delta u^* dx
$$

$$
- (1 + i \rho_1) \int_{\Omega} |u|^2 \Delta u^* dx + (1 + i \rho_2) \int_{\Omega} |u|^4 \Delta u^* dx.
$$

We have, using the boundary condition and the Green formula,

$$
- \int_{\Omega} \frac{\partial u}{\partial t} \Delta u^* dx = \frac{1}{2} \frac{d}{dt} |\nabla u|^2_{L^2},
$$

$$
- \int_{\Omega} [\rho u + (1 + i \rho_1) |u|^2 u - (1 + i \rho_2) |u|^4 u] \Delta u^* dx
$$

$$
= \int_{\Omega} [\rho u + (1 + i \rho_1) |u|^2 u - (1 + i \rho_2) |u|^4 u]' |\nabla u|^2 dx.
$$

Hence

$$
\frac{1}{2} \frac{d}{dt} |\nabla u|^2_{L^2} = -|\Delta u|^2_{L^2} + \int_{\Omega} [\rho u + |u|^2 u - |u|^4 u]' |\nabla u|^2 dx
$$

$$
= -|\Delta u|^2_{L^2} + \int_{\Omega} [\rho u + \frac{3}{2} |u|^2 - 3 |u|^4] |\nabla u|^2 dx.
$$

We also infer from general results on the Dirichlet problem in $\Omega$ on $H^2(\Omega) \cap H^1_0(\Omega)$, and therefore,

$$
|\nabla u|^2_{L^2} \leq \frac{1}{\lambda_1} |\Delta u|^2_{L^2}, \varepsilon_0 = \varepsilon_1 = 1,
$$

Due to

$$
2 \rho + 3 |u|^2 - 6 |u|^4 \leq -\frac{1}{2} |u|^4 + C_3 \quad \text{for some } C_3 > 0,
$$

$$
\frac{d}{dt} |\nabla u|^2_{L^2} \leq C_3 \frac{1}{2} |\nabla u|^2_{L^2} \equiv C_4 |\nabla u|^2_{L^2}.
$$

If $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, then the Gronwall lemma shows that

$$
|\nabla u(t)|^2_{L^2} \leq |\nabla u_0|^2_{L^2} \exp(C_4 t), \forall t > 0.
$$

The same arguments as in $L^2(\Omega)$ can be considered and we can conclude that CGLE possesses a maximal attractor $A$ which is bounded in $H^2(\Omega) \cap H^1_0(\Omega)$, compact and connected in $L^2(\Omega)$. Its basin of attraction is the whole space $L^2(\Omega)$, and $A$ attracts the bounded sets of $L^2(\Omega)$.

**Remark 6.1.** For the periodic boundary condition, we achieve the same result using Hölder’s and Young inequalities instead of the Poincaré inequality.
6.2. **Concluding remarks.** We studied, throughout this paper, the bifurcation and structure of the bifurcated solutions of complex Ginzburg-Landau equations (CGLE). We first consider CGLE with unstable nonlinear cubic term but not with quintic term and achieved supercritical bifurcation under the periodic boundary condition. The solutions bifurcated from the trivial solution $u = 0$ to an attractor $\Sigma_\rho$ when the system parameter $\rho$ passes the first eigenvalue of the linear operator. However, we should restrict our concerns to some parameter region $1 - \frac{\rho}{\rho_0} < 0$. To study outside of the region, we added an stable quintic term, so called cubic-quintic CGLE. We imposed the Dirichlet and periodic boundary conditions on it and we achieved subcritical bifurcations. Since CGLE has the global attractor near $\rho = 0$, we had a saddle node bifurcation point $\lambda_c$. Moreover, we proved that the bifurcated invariant sets are homeomorphic to $S^1$. We also studied steady state bifurcation and concluded that bifurcated invariant sets contain only steady state solutions.

As mentioned in the introduction, CGLE can be obtained by perturbing nonlinear Schrödinger equation (NLS). Therefore, we can study bifurcation theory of NLS, and link these two phase transition phenomena by perturbation of equations. It will be reported elsewhere.

**References**


(JP) DEPARTMENT OF MATHEMATICS & INSTITUTE FOR SCIENTIFIC COMPUTING AND APPLIED MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405

E-mail address: junjupar@indiana.edu